ACCESSING THE FARRELL–TATE COHOMOLOGY
OF DISCRETE GROUPS

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ABSTRACT. We introduce a method to explicitly determine the Farrell–Tate cohomology of discrete groups. We apply this method to the Coxeter triangle and tetrahedral groups as well as to the Bianchi groups, i.e. $\mathrm{PSL}_2(\mathcal{O})$ for $\mathcal{O}$ the ring of integers in an imaginary quadratic number field. We show that the Farrell–Tate cohomology of the Bianchi groups is completely determined by the numbers of conjugacy classes of finite subgroups. In fact, our access to Farrell–Tate cohomology allows us to detach the information about it from geometric models for the Bianchi groups and to express it only with the group structure. Formulae for the numbers of conjugacy classes of finite subgroups have been determined in a thesis of Krämer, in terms of elementary number-theoretic information on $\mathcal{O}$. An evaluation of these formulae for a large number of Bianchi groups is provided numerically in the appendix. Our new insights about the homological torsion allow us to give a conceptual description of the cohomology ring structure of the Bianchi groups.

1. INTRODUCTION

Farrell–Tate cohomology $\widehat{\mathrm{H}}^q$ (which we will by now just call Farrell cohomology) coincides with group homology H^q of groups in all degrees q above their virtual cohomological dimension [7]. So for instance for the Coxeter groups, the Farrell cohomology yields all of the group homology.

In section 2, we will introduce a method of how to explicitly determine the Farrell cohomology, provided that we have enough information about a cell complex on which the discrete group under investigation acts, with cell stabilizers fixing their cells point-wise. This method has also been implemented on the computer [10], which allows us to check the results that we obtain by our arguments. We apply our method to the Coxeter triangle and tetrahedral groups in section 3, and to the Bianchi groups in sections 4 through 6.

We require any discrete group Γ under our study to be provided with a cell complex on which it acts cellularly. We call this a Γ-*cell complex*. Let X be a Γ-cell complex; and let ℓ be a prime number. Denote by S the set of all the cells σ of X, such that there exists an element of order ℓ in the stabilizer of the cell σ. In the case that the stabilizers are finite and fix their cells point-wise, the set S is a Γ-sub-complex of X, and we call it the ℓ-*torsion sub-complex*.

For the Coxeter tetrahedral groups, generated by the reflections on the sides of a tetrahedron in hyperbolic 3-space, we obtain the following. Denote by $\mathcal{D}_\ell$ the dihedral group of order 2ℓ.

Corollary 1 (Corollary to theorem 11.)**.** *Let Γ be a Coxeter tetrahedral group, and $\ell > 2$ be a prime number. Then there is an isomorphism $\mathrm{H}_q(\Gamma;\, \mathbb{Z}/\ell) \cong (\mathrm{H}_q(\mathcal{D}_\ell;\, \mathbb{Z}/\ell))^m$, with m the number of connected components of the orbit space of the ℓ-torsion sub-complex of the Davis complex of Γ.*

We specify the exponent m in the tables in figures 2 through 5.

Results for the Bianchi groups. Denote by $\mathbb{Q}(\sqrt{-m})$, with m a square-free positive integer, an imaginary quadratic number field, and by $\mathcal{O}_{-m}$ its ring of integers. The *Bianchi groups* are the groups $\mathrm{PSL}_2(\mathcal{O}_{-m})$. The Bianchi groups may be considered as a key to the study of a larger class of groups, the *Kleinian* groups, which date back to work of Henri Poincaré [19]. In fact, each non-co-compact arithmetic Kleinian group is commensurable with some Bianchi group [17]. A wealth of information on the Bianchi groups can be found in the monographs [12], [11], [17]. Krämer [16] has determined number-theoretic formulae for the numbers of conjugacy classes of finite subgroups in the Bianchi groups, using numbers of ideal classes in orders of cyclotomic extensions of $\mathbb{Q}(\sqrt{-m})$.

In section 5, we express the homological torsion of the Bianchi groups as a function of these numbers of conjugacy classes. To achieve this, we build on the geometric techniques of [21], which depend on the explicit knowledge of the quotient space of geometric models for the Bianchi groups — like any technique effectively accessing the (co)homology of the Bianchi groups, either directly [25], [29] or via a group presentation [5]. For the Bianchi groups, we can in sections 4 and 5 detach invariants of the group actions from the geometric models, in order to express them only by the group structure itself, in terms of conjugacy classes of finite subgroups, normalizers of the latter, and their interactions. This information is already contained in our reduced torsion sub-complexes.

Not only does this provide us with exact formulae for the homological torsion of the Bianchi groups, the power of which we can see in the numerical evaluations of appendices A.1 and A.2, also it allows us to understand the rôle of the centralizers of the finite subgroups, and this is how in [20], some more fruits of the present results are harvested (in terms of the Chen/Ruan orbifold cohomology of the orbifolds given by the action of the Bianchi groups on complexified hyperbolic space).

Except for the Gaussian and Eisenstein integers, which can easily be treated separately [25], [21], all the rings of integers of imaginary quadratic number fields admit as only units $\{\pm 1\}$. In the latter case, we call $\mathrm{PSL}_2(\mathcal{O})$ a *Bianchi group with units* $\{\pm 1\}$. For the possible types of finite subgroups in the Bianchi groups, see lemma 17 : There are five non-trivial possibilities. In theorem 2, the proof of which we give in section 5, we give a formula expressing precisely how the Farrell cohomology of the Bianchi groups with units $\{\pm 1\}$ depends on the numbers of conjugacy classes of non-trivial finite subgroups of the occurring five types. The main step in order to prove this, is to read off the Farrell cohomology from the reduced torsion sub-complexes.

Krämer's formulae express the numbers of conjugacy classes of the five types of non-trivial finite subgroups in the Bianchi groups, where the symbols in the first row are Krämer's notations for the number of their conjugacy classes:

λ_4	λ_6	μ_2	μ_3	μ_T
$\mathbb{Z}/2$	$\mathbb{Z}/3$	$\mathcal{D}_2$	$\mathcal{S}_3$	$\mathcal{A}_4$

Recall that we can express the homology in degrees above the virtual cohomological dimension of the Bianchi groups by the two Poincaré series — for $\ell = 2$ and $\ell = 3$ — in the dimensions over the field with ℓ elements, of the homology with $\mathbb{Z}/\ell$-coefficients of $\mathrm{PSL}_2(\mathcal{O}_{-m})$,

$$P_m^\ell(t) := \sum_{q > 2}^{\infty} \dim_{\mathbb{F}_\ell} \mathrm{H}_q\left(\mathrm{PSL}_2(\mathcal{O}_{-m}); \mathbb{Z}/\ell\right) t^q,$$

which have been suggested by Grunewald. Further let $P_{\bigcirc}(t) := \frac{-2t^3}{t-1}$, which equals the series $P_m^2(t)$ of the groups $\mathrm{PSL}_2\left(\mathcal{O}_{-m}\right)$ the reduced 2-torsion sub-complex of which is a circle. Denote by

- $P_{\mathcal{D}_2}^*(t) := \frac{-t^3(3t-5)}{2(t-1)^2}$, the Poincaré series over $\dim_{\mathbb{F}_2} \mathrm{H}_q\left(\mathcal{D}_2;\, \mathbb{Z}/2\right) - \frac{3}{2}\dim_{\mathbb{F}_2} \mathrm{H}_q\left(\mathbb{Z}/2;\, \mathbb{Z}/2\right)$
- and by $P_{\mathcal{A}_4}^*(t) := \frac{-t^3(t^3-2t^2+2t-3)}{2(t-1)^2(t^2+t+1)}$, the Poincaré series over

$$\dim_{\mathbb{F}_2} \mathrm{H}_q\left(\mathcal{A}_4;\, \mathbb{Z}/2\right) - \frac{1}{2}\dim_{\mathbb{F}_2} \mathrm{H}_q\left(\mathbb{Z}/2;\, \mathbb{Z}/2\right).$$

In 3-torsion, let $P_{\bullet\!-\!\bullet}(t) := \frac{-t^3(t^2-t+2)}{(t-1)(t^2+1)}$, which equals the series $P_m^3(t)$ for the Bianchi groups the reduced 3-torsion sub-complex of which is a single edge without identifications.

Theorem 2. *For all Bianchi groups with units $\{\pm 1\}$, the homology in degrees above their virtual cohomological dimension is given by the Poincaré series*

$$P_m^2(t) = \left(\lambda_4 - \frac{3\mu_2 - 2\mu_T}{2}\right) P_{\bigcirc}(t) + (\mu_2 - \mu_T)P_{\mathcal{D}_2}^*(t) + \mu_T P_{\mathcal{A}_4}^*(t)$$

and

$$P_m^3(t) = \left(\lambda_6 - \frac{\mu_3}{2}\right) P_{\bigcirc}(t) + \frac{\mu_3}{2} P_{\bullet\!-\!\bullet}(t).$$

Organization of the paper. In section 2, we introduce our method to explicitly determine Farrell cohomology: By reducing the torsion sub-complexes. We apply our method to the Coxeter triangle and tetrahedral groups in section 3. In section 4, we show how to read off the Farrell cohomology of the Bianchi groups from the reduced torsion sub-complexes. We achieve this by showing that for the Bianchi groups, the reduced torsion sub-complexes are homeomorphic to conjugacy classes graphs that we can define without reference to any geometric model. This enables us in section 5 to prove the formulae for the homological torsion of the Bianchi groups in terms of numbers of conjugacy classes of finite subgroups. We use this to establish the structure of the classical cohomology rings of the Bianchi groups in section 6. Krämer has given number-theoretic formulae for these numbers of conjugacy classes, and we evaluate them numerically in appendices A.1 and A.2. Finally, we present some numerical asymptotics on the numbers of conjugacy classes in appendix A.3.

2. Reduction of torsion sub-complexes

Let X be a finite-dimensional cell complex with a cellular action of a discrete group Γ, such that each cell stabilizer fixes its cell point-wise. Let ℓ be a prime such that every non-trivial finite ℓ-subgroup of Γ admits a contractible fixed point set. We keep these requirements on the Γ-action as a general assumption throughout this article. Then, the Γ-equivariant Farrell cohomology of X gives us the ℓ-primary part of the Farrell cohomology of Γ, as follows.

Proposition 3 (Brown [7]). *Under our general assumption, the canonical map*

$$\widehat{H}^*(\Gamma;\, M)_{(\ell)} \to \widehat{H}^*_\Gamma(X;\, M)_{(\ell)}$$

is an isomorphism for any Γ-module M of coefficients.

The classical choice [7] is to take for X the geometric realization of the partially ordered set of non-trivial finite subgroups (respectively, non-trivial elementary Abelian ℓ-subgroups) of Γ, the latter acting by conjugation. The stabilizers are then the normalizers, which in many discrete groups are infinite. And it can impose great computational challenges to determine a group presentation for them. When we want to compute the module $\widehat{H}^*_\Gamma(X;\, M)_{(\ell)}$ subject to proposition 3, at least we must get to know the ℓ-primary part of the Farrell cohomology of these normalizers. The Bianchi groups are an instance that different isomorphism types can occur for this cohomology at different conjugacy classes of elementary Abelian ℓ-subgroups, both for $\ell = 2$ and $\ell = 3$. As the only non-trivial elementary Abelian 3-subgroups in the Bianchi groups are of rank 1, the orbit space $_\Gamma\backslash X$ consists only of one point for each conjugacy class of type $\mathbb{Z}/3$ and a corollary [7] from proposition 3 decomposes the 3-primary part of the Farrell cohomology of the Bianchi groups into the direct product over their normalizers. However, due to the different possible homological types of the normalizers (in fact, two of them occur), the final result remains unclear and subject to tedious case-by-case computations of the normalizers.

In contrast, in the cell complex we are going to develop, the connected components of the orbit space are for the 3-torsion in the Bianchi groups not simple points, but have either the shape ●—● or ⟳. This dichotomy already contains the information about the occurring normalizer.

Definition 4. Let ℓ be a prime number. The *ℓ-torsion sub-complex* of the Γ-cell complex X consists of all the cells of X the stabilizers in Γ of which contain elements of order ℓ.

We are from now on going to require the cell complex X to admit only finite stabilizers in Γ, and we require the action of Γ on the coefficient module M to be trivial. Then obviously only cells from the *ℓ-torsion sub-complex* contribute to $\widehat{H}^*_\Gamma(X;\, M)_{(\ell)}$. We are going to reduce the *ℓ-torsion sub-complex* to one which still carries the Γ-equivariant Farrell cohomology of X, but can have considerably less orbits of cells, can be easier to handle in practice, and, for certain classes of groups, leads us to an explicit structural description of the Farrell cohomology of Γ. The pivotal property of this reduced ℓ-torsion sub-complex will be given in theorem 9.

Condition A. In the ℓ-torsion sub-complex, let σ be a cell of dimension $n - 1$, lying in the boundary of precisely two n-cells τ_1 and τ_2, the latter cells representing two different orbits. Assume further that no higher-dimensional cells of the ℓ-torsion sub-complex touch σ; and that the n-cell stabilizers admit an isomorphism $\Gamma_{\tau_1} \cong \Gamma_{\tau_2}$.

Where this condition is fulfilled in the ℓ-torsion sub-complex, we merge the cells τ_1 and τ_2 along σ and do so for their entire orbits, if and only if they meet the following additional condition. We use Zassenhaus's notion for a finite group to be *ℓ-normal*, if the center of one of its Sylow ℓ-subgroups is the center of every Sylow ℓ-subgroup in which it is contained.

Condition B. Either $\Gamma_{\tau_1} \cong \Gamma_\sigma$,
or Γ_σ is ℓ-normal and $\Gamma_{\tau_1} \cong N_{\Gamma_\sigma}(\text{center}(\text{Sylow}_\ell(\Gamma_\sigma)))$.

Here, we write $\mathrm{N}_{\Gamma_\sigma}$ for taking the normalizer in Γ_σ and Sylow_ℓ for picking an arbitrary Sylow ℓ-subgroup. This is well defined because all Sylow ℓ-subgroups are conjugate.

Lemma 5. *Let $\widetilde{X_\ell}$ be the Γ-complex obtained by orbit-wise merging two n-cells of the ℓ-torsion sub-complex X_ℓ satisfying conditions A and B. Then,*

$$\widehat{\mathrm{H}}_\Gamma^*(\widetilde{X_\ell}; M)_{(\ell)} \cong \widehat{\mathrm{H}}_\Gamma^*(X_\ell; M)_{(\ell)}.$$

For the proof, we will make use of Swan's extension [28, final corollary] to Farrell cohomology of the Second Theorem of Grün [13, Satz 5].

Theorem 6 (Swan)**.** *Let G be a ℓ-normal finite group, and let N be the normalizer of the center of a Sylow ℓ-subgroup of G. Let M be any G-module. Then the inclusion and transfer maps both are isomorphisms between the ℓ-primary components of $\widehat{\mathrm{H}}^*(G; M)$ and $\widehat{\mathrm{H}}^*(N; M)$.*

Proof of lemma 5. Consider the equivariant spectral sequence in Farrell cohomology [7]. On the ℓ-torsion sub-complex, it includes a map

$$\widehat{\mathrm{H}}^*(\Gamma_\sigma; M)_{(\ell)} \xrightarrow{\;d_1^{(n-1),*}|_{\widehat{\mathrm{H}}^*(\Gamma_\sigma; M)_{(\ell)}}\;} \widehat{\mathrm{H}}^*(\Gamma_{\tau_1}; M)_{(\ell)} \oplus \widehat{\mathrm{H}}^*(\Gamma_{\tau_2}; M)_{(\ell)} \, ,$$

which is the diagonal map with blocks the isomorphisms $\widehat{\mathrm{H}}^*(\Gamma_\sigma; M)_{(\ell)} \xrightarrow{\;\cong\;} \widehat{\mathrm{H}}^*(\Gamma_{\tau_i}; M)_{(\ell)} \, ,$ induced by the inclusions $\Gamma_{\tau_i} \hookrightarrow \Gamma_\sigma$. The latter inclusions induce isomorphisms because of theorem 6. If for the orbit of τ_1 or τ_2 we have chosen a representative which is not adjacent to σ, then this isomorphism is composed with the isomorphism induced by conjugation with the element of Γ carrying the cell to one adjacent to σ. Hence, the map $d_1^{(n-1),*}|_{\widehat{\mathrm{H}}^*(\Gamma_\sigma; M)_{(\ell)}}$ has vanishing kernel, and dividing its image out of $\widehat{\mathrm{H}}^*(\Gamma_{\tau_1}; M)_{(\ell)} \oplus \widehat{\mathrm{H}}^*(\Gamma_{\tau_2}; M)_{(\ell)}$ gives the ℓ-primary part $\widehat{\mathrm{H}}^*(\Gamma_{\tau_1 \cup \tau_2}; M)_{(\ell)}$ of the Farrell cohomology of the union $\tau_1 \cup \tau_2$ of the two n-cells. As by condition A no higher-dimensional cells are touching σ, there are no higher degree differentials interfering. $\qquad\square$

The following direct consequence of the Lyndon–Hochschild–Serre spectral sequence allows us to weaken condition B in a way that lemma 5 still holds. Let ℓ be a prime number, and denote by *mod ℓ homology* group homology with $\mathbb{Z}/\ell$-coefficients under the trivial action.

Lemma 7. *Let T be a group with trivial mod ℓ homology, and consider any group extension*

$$1 \to T \to E \to Q \to 1.$$

Then the map $E \to Q$ induces an isomorphism on mod ℓ homology.

This statement may look like a triviality, but it becomes wrong as soon as we exchange the rôles of T and Q in the group extension. In degrees 1 and 2, our claim follows from [7, VII.(6.4)]. In arbitrary degree, it is more or less known and we can proceed through the following easy steps.

Proof. Consider the Lyndon–Hochschild–Serre spectral sequence associated to the group extension, namely

$$E_{p,q}^2 = \mathrm{H}_p(Q; \mathrm{H}_q(T; \mathbb{Z}/\ell)) \text{ converges to } \mathrm{H}_{p+q}(E; \mathbb{Z}/\ell).$$

By our assumption, $H_q(T; \mathbb{Z}/\ell)$ is trivial, so this spectral sequence concentrates in the row $q = 0$, degenerates on the second page and yields isomorphisms

(1) $$H_p(Q; H_0(T; \mathbb{Z}/\ell)) \cong H_p(E; \mathbb{Z}/\ell).$$

As for the modules of co-invariants, we have $((\mathbb{Z}/\ell)_T)_Q \cong (\mathbb{Z}/\ell)_E$ [18], the trivial actions of E and T induce that also the action of Q on the coefficients in $H_0(T; \mathbb{Z}/\ell)$ is trivial. Thus, the isomorphism (1) becomes $H_p(Q; \mathbb{Z}/\ell) \cong H_p(E; \mathbb{Z}/\ell)$.   □

The above lemma directly implies that any extension of two groups both having trivial mod ℓ homology, again has trivial mod ℓ homology.

Our condition B can now be weakened as follows.

Condition B'. The group Γ_σ admits a (possibly trivial) normal subgroup T_σ with trivial mod ℓ homology and with quotient group G_σ; and the group Γ_{τ_1} admits a (possibly trivial) normal subgroup T_τ with trivial mod ℓ homology and with quotient group G_τ making the sequences

$$1 \to T_\sigma \to \Gamma_\sigma \to G_\sigma \to 1 \text{ and } 1 \to T_\tau \to \Gamma_{\tau_1} \to G_\tau \to 1$$

exact and satisfying one of the following.

- Either $G_\tau \cong G_\sigma$,
- or G_σ is ℓ-normal and $G_\tau \cong N_{G_\sigma}(\text{center}(\text{Sylow}_\ell(G_\sigma)))$,
- or both G_σ and G_τ are ℓ-normal and there is a (possibly trivial) group T with trivial mod ℓ homology making the sequence

  $$1 \to T \to N_{G_\sigma}(\text{center}(\text{Sylow}_\ell(G_\sigma))) \to N_{G_\tau}(\text{center}(\text{Sylow}_\ell(G_\tau))) \to 1$$

  exact.

Note that this weaker condition is not yet implemented on the machine, so the above quoted program carries out only the reductions that are subject to condition B.

By a "terminal vertex", we will denote a vertex with no adjacent higher-dimensional cells and precisely one adjacent edge in the quotient space, and by "cutting off" the latter edge, we will mean that we remove the edge together with the terminal vertex from our cell complex.

Definition 8. The *reduced ℓ-torsion sub-complex* associated to a Γ-cell complex X is the cell complex obtained by recursively merging orbit-wise all the pairs of cells satisfying conditions A and B'; and cutting off edges that admit a terminal vertex together with which they satisfy condition B'.

Theorem 9. *There is an isomorphism between the ℓ-primary parts of the Farrell cohomology of Γ and the Γ-equivariant Farrell cohomology of the reduced ℓ-torsion sub-complex.*

Proof. We apply proposition 3 to the cell complex X, and then we apply lemma 5 each time that we orbit-wise merge a pair of cells of the ℓ-torsion sub-complex, or that we cut off an edge. Lemma 7 allows us to replace condition B by condition B' as hypothesis for lemma 5.   □

Remark 10. The computer implementation checks condition B for each pair of cell stabilizers, using a presentation of the latter in terms of matrices, permutation cycles or generators and relators. In the below examples however, we do avoid this case-by-case computation by a general determination of the isomorphism types of pairs of cell stabilizers for which group inclusion induces an isomorphism on mod ℓ homology. The latter method is to be considered as the procedure of preference, because it allows us to deduce statements that hold for the whole class of concerned groups.

3. FARRELL COHOMOLOGY OF THE COXETER TETRAHEDRAL GROUPS

As the Coxeter groups admit a contractible classifying space for proper actions [9], their Farrell cohomology yields all of their group cohomology. So in this section, we make use of this fact to determine the latter. For facts about Coxeter groups, and especially for the Davis complex, we refer to [9]. Recall that the simplest example of a Coxeter group, the dihedral group $\mathcal{D}_n$, is an extension

$$1 \to \mathbb{Z}/n \to \mathcal{D}_n \to \mathbb{Z}/2 \to 1,$$

so we can make use of the original application [30] of Wall's lemma to obtain its mod ℓ homology for prime numbers $\ell > 2$,

$$H_q(\mathcal{D}_n; \mathbb{Z}/\ell) \cong \begin{cases} \mathbb{Z}/\ell, & q = 0, \\ \mathbb{Z}/\gcd(n, \ell), & q \equiv 3 \text{ or } 4 \mod 4, \\ 0, & \text{otherwise.} \end{cases}$$

Theorem 11. *Let $\ell > 2$ be a prime number. Let Γ be a Coxeter group admitting a Coxeter system with at most four generators, and relator orders not divisible by ℓ^2. Let $Z(\ell)$ be the ℓ-torsion sub-complex of the Davis complex of Γ. If $Z(\ell)$ is at most one-dimensional and its orbit space contains no loop nor bifurcation, then the $\mod \ell$ homology of Γ is isomorphic to $(H_q(\mathcal{D}_\ell; \mathbb{Z}/\ell))^m$, with m the number of connected components of the orbit space of $Z(\ell)$.*

The conditions of this theorem are for instance fulfilled by the Coxeter tetrahedral groups; we specify the exponent m for them in the tables in figures 2 through 5. In order to prove theorem 11, we lean on the following intermediary statement.

Lemma 12. *Let $\ell > 2$ be a prime number; and let Γ_σ be a finite Coxeter group with $n \leqslant 4$ generators. If Γ_σ is not a direct product of two dihedral groups and not associated to the Coxeter diagram $\mathbf{F}_4$ or $\mathbf{H}_4$, then condition B' is fulfilled for the triple consisting of ℓ, the group Γ_σ and any of its Coxeter subgroups Γ_{τ_1} with $(n-1)$ generators that contains ℓ-torsion elements.*

Proof. The dihedral groups admit only Coxeter subgroups with two elements, so without ℓ-torsion. There are only finitely many other isomorphism types of irreducible finite Coxeter groups with at most four generators, specified by the Coxeter diagrams

$\mathbf{A}_1$	$\mathbf{A}_3$	$\mathbf{A}_4$	$\mathbf{B}_3$	$\mathbf{B}_4$	$\mathbf{D}_4$	$\mathbf{H}_3$
•	•—•—•	•—•—•—•	•—$\overset{4}{—}$•—•	•—$\overset{4}{—}$•—•—•	(Y-shape)	•—$\overset{5}{—}$•—•

on which we can check the condition case by case.

  $\mathbf{A}_1$. The symmetric group $\mathcal{S}_2$ admits no Coxeter subgroups.
  $\mathbf{A}_3$. The symmetric group $\mathcal{S}_4$ is 3-normal; and its Sylow-3-subgroups are of type $\mathbb{Z}/3$, so they are identical to their center. Their normalizers in $\mathcal{S}_4$ match the Coxeter subgroups of type $\mathcal{D}_3$ that one obtains by omitting one of the generators of $\mathcal{S}_4$ at an end of its Coxeter diagram. The other possible Coxeter subgroup type is $(\mathbb{Z}/2)^2$, obtained by omitting the middle generator in this diagram, and contains no 3-torsion.

A$_4$. The Coxeter subgroups with three generators in the symmetric group $\mathcal{S}_5$ are $\mathcal{D}_3 \times \mathbb{Z}/2$ and $\mathcal{S}_4$, so we only need to consider 3-torsion. The group $\mathcal{S}_5$ is 3-normal; the normalizer of the center of any of its Sylow-3-subgroups is of type $\mathcal{D}_3 \times \mathbb{Z}/2$. So for the Coxeter subgroup $\mathcal{S}_4$, we use the normalizer $\mathcal{D}_3$ of its Sylow-3-subgroup $\mathbb{Z}/3$; and apply the last option of condition B'.

B$_3$. We apply lemma 7 to the Coxeter group $(\mathbb{Z}/2)^3 \rtimes \mathcal{S}_3$, and retain only $\mathcal{S}_3$, which is isomorphic to the only Coxeter subgroup admitting 3-torsion.

B$_4$. The Coxeter subgroups with three generators are of type $\mathcal{S}_4$, $\mathbb{Z}/2 \times \mathcal{D}_3$, $\mathcal{D}_4 \times \mathbb{Z}/2$ or $(\mathbb{Z}/2)^3 \rtimes \mathcal{S}_3$, thus for the three of them containing 3-torsion, we use the above methods to relate them to $\mathcal{D}_3$. The Coxeter group $(\mathbb{Z}/2)^4 \rtimes \mathcal{S}_4$ is 3-normal; its Sylow-3-subgroup is of type $\mathbb{Z}/3$ and admits a normalizer N fitting into the exact sequence

$$1 \to (\mathbb{Z}/2)^2 \to N \to \mathcal{D}_3 \to 1.$$

D$_4$. From the Coxeter diagram, we see that the Coxeter subgroups with three generators are $(\mathbb{Z}/2)^3$ and $\mathcal{S}_4$. So we only need to compare with the 3-torsion of $\mathcal{S}_4$. For this purpose, we apply lemma 7 to the Coxeter group $(\mathbb{Z}/2)^3 \rtimes \mathcal{S}_4$.

H$_3$. The symmetry group Icos_{120} of the icosahedron splits as a direct product $\mathbb{Z}/2 \times \mathcal{A}_5$, so by lemma 7, we can for all primes $\ell > 2$ make use of the alternating group $\mathcal{A}_5$ as the quotient group in condition B'. The primes other than 2, at which the homology of $\mathcal{A}_5$ admits torsion, are 3 and 5. So now let ℓ be 3 or 5. Then the group $\mathcal{A}_5$ is ℓ-normal; and its Sylow-ℓ-subgroups are of type $\mathbb{Z}/\ell$, so they are identical to their center. Their normalizers in $\mathcal{A}_4$ are of type $\mathcal{D}_\ell$. From the Coxeter diagram, we see that this is the only Coxeter subgroup type with two generators that contains ℓ-torsion.

The case where we have a direct product of the one-generator Coxeter group $\mathbb{Z}/2$ with one of the above groups, is already absorbed by condition B'. $\qquad \square$

Proof of theorem 11. The Davis complex is a finite-dimensional cell complex with a cellular action of the Coxeter group Γ with respect to which it is constructed, such that each cell stabilizer fixes its cell point-wise. Also, it admits the property that the fixed point sets of the finite subgroups of Γ are acyclic [9]. Thus by proposition 3, the Γ-equivariant Farrell cohomology of the Davis complex gives us the ℓ-primary part of the Farrell cohomology of Γ. As the 3-torsion sub-complex for the group generated by the Coxeter diagram $\mathbf{F}_4$ (the symmetry group of the 24-cell) and the 3- and 5-torsion sub-complexes for the group generated by the Coxeter diagram $\mathbf{H}_4$ (the symmetry group of the 600-cell) as well as the ℓ-torsion sub-complex of a direct product of two dihedral groups with ℓ-torsion all contain 2-cells, we are either in the case where the ℓ-torsion sub-complex is trivial or in the case in which we suppose to be from now on, namely where Γ is not one of the groups just mentioned. Then all the finite Coxeter subgroups of Γ fulfill the hypothesis of lemma 12, and hence all pairs of a vertex stabilizer and the stabilizer of an adjacent edge satisfy condition B'. By the assumptions on $Z(\ell)$, also condition A is fulfilled for any pair of adjacent edges in $Z(\ell)$. Hence, every connected component of the reduced ℓ-torsion sub-complex is a single vertex. From recursive use of lemma 12 and the assumption that the relator orders are not divisible by ℓ^2, we see that the stabilizer of the latter vertex has the mod ℓ homology of $\mathcal{D}_\ell$. Theorem 9 now yields our claim. $\qquad \square$

Let us determine the exponent m of theorem 11 for some classes of examples. The *Coxeter triangle groups* are given by the presentation

$$\langle\, a, b, c \mid a^2 = b^2 = c^2 = (ab)^p = (bc)^q = (ca)^r = 1 \,\rangle,$$

where $2 \leqslant p, q, r \in \mathbb{N}$ and $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} \leq 1$.

Proposition 13. *For any prime $\ell > 2$, the* mod ℓ *homology of a Coxeter triangle group is given as the direct sum over the* mod ℓ *homology of the dihedral groups $\mathcal{D}_p$, $\mathcal{D}_q$ and $\mathcal{D}_r$.*

Proof. The quotient space of the Davis complex of a Coxeter triangle group can be realized as the barycentric subdivision of an Euclidean or hyperbolic triangle with interior angles $\frac{\pi}{p}$, $\frac{\pi}{q}$ and $\frac{\pi}{r}$, and a, b and c acting as reflections through the corresponding sides. We obtain this triangle
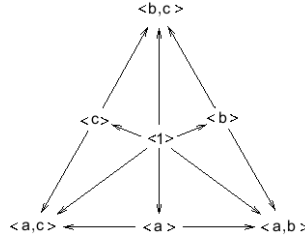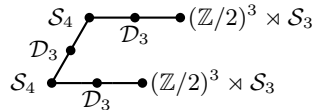

FIGURE 1. Quotient of the Davis complex for a triangle group (diagram reprinted with the kind permission of Sanchez-Garcia [24]).

by realizing the partially ordered set (where arrows stand for inclusions) of Figure 1. The whole Davis complex of the Coxeter triangle groups is then given as a tessellation of the Euclidean or hyperbolic plane by these triangles. The quotient space of the ℓ-torsion sub-complex then consists of one vertex for each of the dihedral groups $\mathcal{D}_p$, $\mathcal{D}_q$ and $\mathcal{D}_r$ which contain an element of order ℓ. Theorem 9 now yields the result. $\qquad\square$

3.1. **Results for the Coxeter tetrahedral groups.** Consider the groups that are generated by the reflections on the four sides of a tetrahedron in hyperbolic 3-space, such that the images of the tetrahedron tessellate the latter. Up to isomorphism, there are only thirty-two such groups [11]; and we call them the Coxeter tetrahedral groups $CT(n)$, with n running from 1 through 32.

Proposition 14. *For all prime numbers $\ell > 2$, the* mod ℓ *homology of all the Coxeter tetrahedral groups is specified in the tables in figures 2 through 5 in all the cases where it is non-trivial.*

Proof. Consider the Coxeter tetrahedral group $CT(25)$, generated by the Coxeter diagram . Then the Davis complex of $CT(25)$ has a strict fundamental domain isomorphic to the barycentric subdivision of the hyperbolic tetrahedron the reflections on the sides of which generate $CT(25)$ geometrically. A strict fundamental domain for the action on the 3-torsion sub-complex is then the graph

Name	Coxeter graph	3−torsion subcomplex quotient	reduced 3−torsion subcomplex quotient	$H_q(CT(m); \mathbb{Z}/3)$
$CT(1)$		$(\mathbb{Z}/2)^3 \rtimes \mathcal{S}_3 \underset{\mathcal{D}_3}{\rule{1.5em}{0.4pt}} \mathcal{S}_3 \times \mathbb{Z}/2$	$\bullet\mathcal{D}_3$	$H_q(\mathcal{D}_3; \mathbb{Z}/3)$
$CT(2)$		$(\mathbb{Z}/2)^3 \rtimes \mathcal{S}_3 \underset{\mathcal{D}_3}{\rule{1.5em}{0.4pt}} (\mathbb{Z}/2)^3 \rtimes \mathcal{S}_3$	$\bullet\mathcal{D}_3$	$H_q(\mathcal{D}_3; \mathbb{Z}/3)$
$CT(3)$		$\mathcal{D}_3 \diagup (\mathbb{Z}/2)^3 \rtimes \mathcal{S}_3$, $\mathcal{S}_4 \underset{\mathcal{D}_3}{\rule{1.5em}{0.4pt}} (\mathbb{Z}/2)^3 \rtimes \mathcal{S}_3$	$\bullet\mathcal{D}_3$	$H_q(\mathcal{D}_3; \mathbb{Z}/3)$
$CT(7)$		$\mathcal{D}_6 \rule{1.5em}{0.4pt} \mathcal{D}_6 \times \mathbb{Z}/2$, $\mathcal{D}_3 \rule{1.5em}{0.4pt} (\mathbb{Z}/2)^3 \rtimes \mathcal{S}_3$	$\bullet\mathcal{D}_6 \bullet \mathcal{D}_3$	$(H_q(\mathcal{D}_3; \mathbb{Z}/3))^2$
$CT(8)$		$\mathcal{D}_3 \rule{1.5em}{0.4pt} \mathcal{D}_3 \times \mathbb{Z}/2$, $\mathcal{D}_3 \underset{\mathcal{S}_4 \; \mathcal{D}_3 \; \mathcal{S}_4}{\rule{3em}{0.4pt}} \mathcal{D}_3$	$\bullet\mathcal{D}_3 \bullet \mathcal{D}_3$	$(H_q(\mathcal{D}_3; \mathbb{Z}/3))^2$
$CT(9)$		$\mathcal{D}_3 \rule{1.5em}{0.4pt} \mathcal{D}_3 \times \mathbb{Z}/2$, $\mathcal{D}_3 \rule{1.5em}{0.4pt} (\mathbb{Z}/2)^3 \rtimes \mathcal{S}_3$, $\mathcal{D}_3 \rule{1.5em}{0.4pt} (\mathbb{Z}/2)^3 \rtimes \mathcal{S}_3$	$\bullet\mathcal{D}_3 \bullet \mathcal{D}_3 \bullet \mathcal{D}_3$	$(H_q(\mathcal{D}_3; \mathbb{Z}/3))^3$
$CT(10)$		$\mathcal{D}_3 \diagup \mathcal{D}_6 \rule{1.5em}{0.4pt} \mathcal{D}_6 \times \mathbb{Z}/2$, $\mathcal{S}_4 \; \mathcal{D}_3 \rule{1.5em}{0.4pt} \mathcal{D}_3 \times \mathbb{Z}/2$	$\bullet\mathcal{D}_6 \bullet \mathcal{D}_3$	$(H_q(\mathcal{D}_3; \mathbb{Z}/3))^2$
$CT(11)$		$\mathcal{D}_3 \underset{\mathcal{D}_3 \times \mathbb{Z}/2}{\rule{3em}{0.4pt}} \mathcal{D}_3 \quad \bullet \mathcal{D}_6$	$\bullet\mathcal{D}_6 \bullet \mathcal{D}_3$	$(H_q(\mathcal{D}_3; \mathbb{Z}/3))^2$
$CT(12)$		six copies of $\bullet \mathcal{D}_3$	six copies of $\bullet \mathcal{D}_3$	$(H_q(\mathcal{D}_3; \mathbb{Z}/3))^6$
$CT(13)$		$\bullet\mathcal{D}_6 \; \bullet\mathcal{D}_3 \; \bullet\mathcal{D}_3$, $\mathcal{D}_3 \rule{1.5em}{0.4pt} \mathcal{D}_3 \times \mathbb{Z}/2$	$\bullet\mathcal{D}_6 \bullet \mathcal{D}_3 \bullet \mathcal{D}_3 \bullet \mathcal{D}_3$	$(H_q(\mathcal{D}_3; \mathbb{Z}/3))^4$

FIGURE 2. 3-torsion sub-complexes of the Coxeter tetrahedral groups $CT(1)$ through $CT(13)$, in the cases where they are non-trivial.

where the labels specify the isomorphism types of the stabilizers, namely the dihedral group $\mathcal{D}_3$, which also stabilizes the edges, the symmetric group $\mathcal{S}_4$ and the semi-direct product $(\mathbb{Z}/2)^2 \rtimes \mathcal{S}_3$. The ℓ-torsion sub-complexes for all greater primes ℓ are empty. By theorem 11, we can reduce the 3-torsion sub-complex to a single vertex and obtain $H_*(CT(25); \mathbb{Z}/3) \cong H_*(\mathcal{D}_3; \mathbb{Z}/3)$. For the other Coxeter tetrahedral groups, we proceed analogously. $\qquad\square$

The entries in the tables in figures 2 through 5 have additionally been checked on the machine [10].

Name	Coxeter graph	3−torsion subcomplex quotient	reduced 3−torsion subcomplex quotient	$H_q(CT(m); \mathbb{Z}/3)$
$CT(14)$		$\mathcal{D}_6 \bullet\!\!-\!\!\bullet \mathcal{D}_6 \times \mathbb{Z}/2$ $\mathcal{D}_6 \bullet\!\!-\!\!\bullet \mathcal{D}_6 \times \mathbb{Z}/2$ $\quad \bullet \mathcal{D}_3$	$\bullet \mathcal{D}_6 \bullet \mathcal{D}_6 \bullet \mathcal{D}_3$	$(H_q(\mathcal{D}_3; \mathbb{Z}/3))^3$
$CT(15)$		$\mathcal{D}_3 \bullet\!\!-\!\!\bullet \mathcal{D}_3 \times \mathbb{Z}/2$ $\mathcal{D}_3 \bullet\!\!-\!\!\bullet \mathcal{D}_3 \times \mathbb{Z}/2$ $\quad \bullet \mathcal{D}_6$	$\bullet \mathcal{D}_6 \bullet \mathcal{D}_3 \bullet \mathcal{D}_3$	$(H_q(\mathcal{D}_3; \mathbb{Z}/3))^3$
$CT(16)$		$\mathcal{D}_3 \bullet\!\!-\!\!\overset{\mathcal{S}_4}{\bullet}\!\!-\!\!\bullet \mathcal{D}_3$ $\mathcal{D}_3 \bullet\!\!-\!\!\underset{\mathcal{S}_4}{\bullet}\!\!-\!\!\bullet \mathcal{D}_3$ $\quad \bullet \mathcal{D}_3$	$\bullet \mathcal{D}_3 \bullet \mathcal{D}_3 \bullet \mathcal{D}_3$	$(H_q(\mathcal{D}_3; \mathbb{Z}/3))^3$
$CT(17)$		$\bullet \mathcal{D}_6 \bullet \mathcal{D}_6 \bullet \mathcal{D}_3 \bullet \mathcal{D}_3$	$\bullet \mathcal{D}_6 \bullet \mathcal{D}_6 \bullet \mathcal{D}_3 \bullet \mathcal{D}_3$	$(H_q(\mathcal{D}_3; \mathbb{Z}/3))^4$
$CT(18)$		$(\mathbb{Z}/2)^3 \rtimes \mathcal{S}_3 \overset{\mathcal{D}_3}{\bullet\!\!-\!\!\bullet} (\mathbb{Z}/2)^3 \rtimes \mathcal{S}_3$ $(\mathbb{Z}/2)^3 \rtimes \mathcal{S}_3 \underset{\mathcal{D}_3}{\bullet\!\!-\!\!\bullet} (\mathbb{Z}/2)^3 \rtimes \mathcal{S}_3$	$\bullet \mathcal{D}_3 \bullet \mathcal{D}_3$	$(H_q(\mathcal{D}_3; \mathbb{Z}/3))^2$
$CT(19)$		$(\mathbb{Z}/2)^3 \rtimes \mathcal{S}_3 \overset{\mathcal{D}_3}{\bullet\!\!-\!\!\bullet} \mathrm{Icos}_{120}$	$\bullet \mathcal{D}_3$	$H_q(\mathcal{D}_3; \mathbb{Z}/3)$
$CT(20)$		$\mathcal{D}_3 \bullet\!\!-\!\!\bullet \mathrm{Icos}_{120}$ $\underset{\mathcal{S}_4 \; \mathcal{D}_3}{\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet} \mathrm{Icos}_{120}$	$\bullet \mathcal{D}_3$	$H_q(\mathcal{D}_3; \mathbb{Z}/3)$
$CT(21)$		$\mathrm{Icos}_{120} \overset{\mathcal{D}_3}{\bullet\!\!-\!\!\bullet} \mathrm{Icos}_{120}$ $\mathrm{Icos}_{120} \underset{\mathcal{D}_3}{\bullet\!\!-\!\!\bullet} \mathrm{Icos}_{120}$	$\bullet \mathcal{D}_3 \bullet \mathcal{D}_3$	$(H_q(\mathcal{D}_3; \mathbb{Z}/3))^2$
$CT(22)$		$\mathrm{Icos}_{120} \overset{\mathcal{D}_3}{\bullet\!\!-\!\!\bullet} \mathcal{D}_3 \times \mathbb{Z}/2$ $\mathrm{Icos}_{120} \underset{\mathcal{D}_3}{\bullet\!\!-\!\!\bullet} \mathcal{D}_3 \times \mathbb{Z}/2$	$\bullet \mathcal{D}_3 \bullet \mathcal{D}_3$	$(H_q(\mathcal{D}_3; \mathbb{Z}/3))^2$
$CT(23)$		$\overset{\mathcal{S}_4}{\bullet}\!\!-\!\!\overset{\mathcal{D}_3}{\bullet}\!\!-\!\!\bullet \mathrm{Icos}_{120}$ $\mathcal{D}_3$ $\underset{\mathcal{S}_4 \quad \mathcal{D}_3}{\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet} \mathrm{Icos}_{120}$	$\bullet \mathcal{D}_3$	$H_q(\mathcal{D}_3; \mathbb{Z}/3)$
$CT(24)$		$\mathrm{Icos}_{120} \underset{\mathcal{D}_3}{\bullet\!\!-\!\!\bullet} \mathrm{Icos}_{120}$	$\bullet \mathcal{D}_3$	$H_q(\mathcal{D}_3; \mathbb{Z}/3)$
$CT(25)$		$\overset{\mathcal{S}_4}{\bullet}\!\!-\!\!\overset{\mathcal{D}_3}{\bullet}\!\!-\!\!\bullet (\mathbb{Z}/2)^3 \rtimes \mathcal{S}_3$ $\mathcal{D}_3$ $\underset{\mathcal{S}_4 \quad \mathcal{D}_3}{\bullet\!\!-\!\!\bullet\!\!-\!\!\bullet} (\mathbb{Z}/2)^3 \rtimes \mathcal{S}_3$	$\bullet \mathcal{D}_3$	$H_q(\mathcal{D}_3; \mathbb{Z}/3)$
$CT(26)$		$\mathrm{Icos}_{120} \overset{\mathcal{D}_3}{\bullet\!\!-\!\!\bullet} (\mathbb{Z}/2)^3 \rtimes \mathcal{S}_3$ $\mathrm{Icos}_{120} \underset{\mathcal{D}_3}{\bullet\!\!-\!\!\bullet} (\mathbb{Z}/2)^3 \rtimes \mathcal{S}_3$	$\bullet \mathcal{D}_3 \bullet \mathcal{D}_3$	$(H_q(\mathcal{D}_3; \mathbb{Z}/3))^2$
$CT(27)$		$\mathcal{D}_3 \bullet\!\!-\!\!\bullet \mathcal{D}_3 \times \mathbb{Z}/2$ $\mathcal{D}_3 \bullet\!\!-\!\!\bullet \mathrm{Icos}_{120}$ $\mathcal{D}_3 \bullet\!\!-\!\!\bullet \mathrm{Icos}_{120}$	$\bullet \mathcal{D}_3 \bullet \mathcal{D}_3 \bullet \mathcal{D}_3$	$(H_q(\mathcal{D}_3; \mathbb{Z}/3))^3$

FIGURE 3. 3-torsion sub-complexes of the Coxeter tetrahedral groups $CT(14)$ through $CT(27)$.

Name	Coxeter graph	3−torsion subcomplex quotient	reduced 3−torsion subcomplex quotient	$H_q(CT(m); \mathbb{Z}/3)$
$CT(28)$	•—5—•—6—•	$\mathcal{D}_6$ •——• $\mathcal{D}_6 \times \mathbb{Z}/2$ $\mathcal{D}_3$ •——• Icos_{120}	•$\mathcal{D}_6$ • $\mathcal{D}_3$	$(H_q(\mathcal{D}_3; \mathbb{Z}/3))^2$
$CT(29)$	6 ▢ 5	$\mathcal{D}_3$ •——• Icos_{120} •$\mathcal{D}_6$ $\mathcal{D}_3$ •——• Icos_{120}	•$\mathcal{D}_6$ • $\mathcal{D}_3$ • $\mathcal{D}_3$	$(H_q(\mathcal{D}_3; \mathbb{Z}/3))^3$
$CT(30)$	6 ▢ 4	$\mathcal{D}_3$ •——• $(\mathbb{Z}/2)^3 \rtimes \mathcal{S}_3$ •$\mathcal{D}_6$ $\mathcal{D}_3$ •——• $(\mathbb{Z}/2)^3 \rtimes \mathcal{S}_3$	•$\mathcal{D}_6$ • $\mathcal{D}_3$ • $\mathcal{D}_3$	$(H_q(\mathcal{D}_3; \mathbb{Z}/3))^3$
$CT(31)$	4 ▢ 4 4	$(\mathbb{Z}/2)^3 \rtimes \mathcal{S}_3$ •—$\mathcal{D}_3$—• $(\mathbb{Z}/2)^3 \rtimes \mathcal{S}_3$	•$\mathcal{D}_3$	$H_q(\mathcal{D}_3; \mathbb{Z}/3)$
$CT(32)$	▢ 6	•$\mathcal{D}_6$ $\mathcal{D}_3$ •—$\mathcal{S}_4$—•—$\mathcal{D}_3$—•—$\mathcal{S}_4$—• $\mathcal{D}_3$	•$\mathcal{D}_6$ • $\mathcal{D}_3$	$(H_q(\mathcal{D}_3; \mathbb{Z}/3))^2$

FIGURE 4. 3-torsion sub-complexes of the Coxeter tetrahedral groups $CT(28)$ through $CT(32)$.

Name and Coxeter graph	5−torsion subcomplex quotient	reduced 5−torsion subcomplex quotient	$H_q(CT(m); \mathbb{Z}/5)$
$CT(19)$, $CT(28)$ •—4—•—5—• •—5—•—6—•	Icos_{120} •—$\mathcal{D}_5$—• $\mathcal{D}_5 \times \mathbb{Z}/2$	•$\mathcal{D}_5$	$H_q(\mathcal{D}_5; \mathbb{Z}/5)$
$CT(20)$, $CT(22)$, $CT(23)$, $CT(26)$, $CT(27)$, $CT(29)$	Icos_{120} •—$\mathcal{D}_5$—• Icos_{120}	•$\mathcal{D}_5$	$H_q(\mathcal{D}_5; \mathbb{Z}/5)$
$CT(21)$ 5 ▢ 5	Icos_{120} •—$\mathcal{D}_5$—• Icos_{120} Icos_{120} •—$\mathcal{D}_5$—• Icos_{120}	•$\mathcal{D}_5$ • $\mathcal{D}_5$	$(H_q(\mathcal{D}_5; \mathbb{Z}/5))^2$
$CT(24)$ •—5—•—5—•	Icos_{120} •—$\mathcal{D}_5$—• $\mathcal{D}_5 \times \mathbb{Z}/2$ Icos_{120} •—$\mathcal{D}_5$—• $\mathcal{D}_5 \times \mathbb{Z}/2$	•$\mathcal{D}_5$ • $\mathcal{D}_5$	$(H_q(\mathcal{D}_5; \mathbb{Z}/5))^2$

FIGURE 5. 5-torsion sub-complexes of the Coxeter tetrahedral groups, in the cases where they are non-trivial.

4. The conjugacy classes of finite order elements in the Bianchi groups

The groups $\mathrm{SL}_2(\mathcal{O}_{-m})$ act in a natural way on hyperbolic three-space, which is isomorphic to the symmetric space $\mathrm{SL}_2(\mathbb{C})/\mathrm{SU}(2)$ associated to them. The kernel of this action is the center $\{\pm 1\}$ of the groups. Thus it is useful to study the quotient of $\mathrm{SL}_2(\mathcal{O}_{-m})$ by its center, namely $\mathrm{PSL}_2(\mathcal{O}_{-m})$.

Let $\Gamma = \mathrm{PSL}_2(\mathcal{O}_{-m})$ be a Bianchi group. Then any element of Γ fixing a point inside real hyperbolic 3-space $\mathcal{H}_{\mathbb{R}}^3$ acts as a rotation of finite order. Let X be the refined cellular complex obtained from the action of Γ on hyperbolic 3-space as described in [21], namely we subdivide $\mathcal{H}_{\mathbb{R}}^3$ until the stabilizer in Γ of any cell σ fixes σ point-wise. We achieve this by computing Bianchi's fundamental polyhedron for the action of Γ, taking as preliminary set of 2-cells its facets lying on the Euclidean hemispheres and vertical planes of the upper-half space model for $\mathcal{H}_{\mathbb{R}}^3$, and then subdividing along the rotation axes of the elements of Γ. Let ℓ be a prime number.

It is well-known that if γ is an element of Bianchi group of finite order n, then n must be 1, 2, 3, 4 or 6, because γ has eigenvalues ρ and $\overline{\rho}$ with ρ a primitive n-th root of unity and the trace of γ is $\rho + \overline{\rho} \in \mathcal{O}_{-m} \cap \mathbb{R} = \mathbb{Z}$. For ℓ being one of the two occurring primes 2 and 3, this sub-complex is a finite graph, because the cells of dimension greater than 1 are trivially stabilized in the refined cellular complex. We reduce this sub-complex with the procedure of [21], which consists in taking the pairs of edges with a common endpoint such that no further edge is adjacent to this endpoint, and replacing them together with this endpoint by a single edge.

We construct the following graph purely group-theoretically in order to locate conjugacy classes of finite groups on the reduced torsion sub-complex. Let ℓ be a prime number. For a circle to become a graph, we identify the two endpoints of a single edge.

Definition 15. *The ℓ-conjugacy classes graph of an arbitrary group Γ is given by the following construction.*

- We take as vertices the conjugacy classes of finite subgroups G of Γ containing elements γ of order ℓ such that the normalizer of $\langle \gamma \rangle$ in G is not $\langle \gamma \rangle$ itself.
- We connect two vertices by an edge if and only if they admit representatives sharing a common subgroup of order ℓ.
- For every pair of subgroups of order ℓ in G, which are conjugate in Γ but not in G, we draw a circle attached to the vertex labeled by G.
- For every conjugacy class of subgroups of order ℓ which are not properly contained in any finite subgroup of Γ, we add a disjoint circle.

Theorem 16. *Let Γ be any Bianchi group with units $\{\pm 1\}$ and ℓ any prime number. Then the ℓ-conjugacy classes graph and the reduced ℓ-torsion sub-complex of Γ are isomorphic graphs.*

The rest of this section will be devoted to the proof of this theorem. The first ingredient is the following classification of Felix Klein [14].

Lemma 17 (Klein). *The finite subgroups in $\mathrm{PSL}_2(\mathcal{O})$ are exclusively of isomorphism types the cyclic groups of orders one, two and three, the Klein four-group $\mathcal{D}_2 \cong \mathbb{Z}/2 \times \mathbb{Z}/2$, the symmetric group $\mathcal{S}_3$ and the alternating group $\mathcal{A}_4$.*

Recall the following lemma from [21].

Lemma 18. *Let v be a non-singular vertex in the refined cell complex. Then the number $\mathbf{n}$ of orbits of edges in the refined cell complex adjacent to v, with stabilizer in $\mathrm{PSL}_2(\mathcal{O}_{-m})$ isomorphic*

to $\mathbb{Z}/\ell$, is given as follows for $\ell = 2$ and $\ell = 3$.

Isomorphism type of the vertex stabiliser	$\{1\}$	$\mathbb{Z}/2$	$\mathbb{Z}/3$	$\mathcal{D}_2$	$\mathcal{S}_3$	$\mathcal{A}_4$
n for $\ell = 2$	0	2	0	3	2	1
n for $\ell = 3$	0	0	2	0	1	2.

Now we investigate the associated normalizer groups. Straight-forward verification using the multiplication tables of the concerned finite groups yields the following.

Lemma 19. *Let G be a finite subgroup of $\mathrm{PSL}_2(\mathcal{O}_{-m})$. Then the type of the normalizer of any subgroup of type $\mathbb{Z}/\ell$ in G is given as follows for $\ell = 2$ and $\ell = 3$, where we print only cases with existing subgroup of type $\mathbb{Z}/\ell$.*

Isomorphism type of G	$\{1\}$	$\mathbb{Z}/2$	$\mathbb{Z}/3$	$\mathcal{D}_2$	$\mathcal{S}_3$	$\mathcal{A}_4$
normaliser of $\mathbb{Z}/2$		$\mathbb{Z}/2$		$\mathcal{D}_2$	$\mathbb{Z}/2$	$\mathcal{D}_2$
normaliser of $\mathbb{Z}/3$			$\mathbb{Z}/3$		$\mathcal{S}_3$	$\mathbb{Z}/3$.

The final ingredient in the proof of theorem 16 is the following.

Theorem 20. *There is a natural bijection between conjugacy classes of subgroups of $\mathrm{PSL}_2(\mathcal{O}_{-m})$ of order ℓ and edges of the reduced ℓ-torsion sub-complex. It is given by considering the stabilizer of a representative edge in the refined cell complex.*

In order to prove the latter theorem, we need several lemmata, and we establish them now.

Lemma 21. *Consider two adjacent edges E, E' of the non-reduced torsion sub-complex. Then for any representative e of E, there is an adjacent representative e' of E' on the same geodesic line as e.*

Proof. Consider the element $\gamma \in \Gamma$ identifying the end v of e with the origin $\gamma(v)$ of e'. As E and E' are distinct in the orbit space, γ^{-1} cannot send e' onto e.
Assume that the stabilizer of v is of isomorphism type $\mathbb{Z}/\ell\mathbb{Z}$. Then as the edge $\gamma^{-1}(e')$ is pointwise fixed by Γ, its stabilizer must contain the rotation with axis passing through e. Hence $\gamma^{-1}(e')$ is adjacent to e and on the same geodesic line.
If the stabilizer of v is of isomorphism type $\mathcal{D}_2$, $\mathcal{S}_3$ or $\mathcal{A}_4$, there are at most two orbits of ℓ-torsion stabilized edges adjacent to v. So there is an element α in the stabilizer of v such that $\alpha\gamma^{-1}(e')$ is adjacent to e and on the same geodesic line. $\qquad\square$

Corollary 22. *Any edge of the reduced torsion sub-complex can be represented by a chain of edges on the intersection of one geodesic line with a strict fundamental domain for Γ in $\mathcal{H}$.*

Proof. Consider the chain of edges in the torsion sub-complex that are reduced to the given edge of the reduced torsion sub-complex. The representatives in $\mathcal{H}$ of the edges on this chain lie on pairwise different orbits because the torsion sub-complex is a sub-complex of the orbit space $_\Gamma\backslash\mathcal{H}$. Now we start with a representative e for the edge sharing its origin with the reduced edge, and use lemma 21 to obtain an adjacent edge e' on the same geodesic line, representing the adjacent edge of the torsion sub-complex. We proceed this way, assigning the role of e to e' and so on, until we have reached the end of the reduced edge. $\qquad\square$
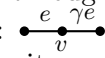
Corollary 23. *Any edge of the reduced torsion sub-complex admits only representatives with stabilizer in the same conjugacy class.*

Lemma 24. *Let α and γ be elements of $\mathrm{PSL}_2(\mathbb{C})$. Then the fixed point set in $\mathcal{H}$ of α is identified by γ with the fixed point set of $\gamma\alpha\gamma^{-1}$.*

Proof. One immediately checks that any fixed point $x \in \mathcal{H}$ of α induces the fixed point $\gamma(x)$ of $\gamma\alpha\gamma^{-1}$. As $\mathrm{PSL}_2(\mathbb{C})$ acts by isometries, the whole fixed point sets are identified. $\square$

Lemma 25. *Let $v \in \mathcal{H}^3_{\mathbb{R}}$ be a vertex with stabilizer in Γ of type $\mathcal{D}_2$ or $\mathcal{A}_4$. Let $\gamma \in \Gamma$ be a rotation of order 2 around an edge e adjacent to v.*
Then the centralizer $C_\Gamma(\gamma)$ reflects $\mathcal{H}^\gamma$ — which is the geodesic line through e — onto itself at v.

Proof. Denote by Γ_v the stabilizer of the vertex v. In the case that Γ_v is of type $\mathcal{D}_2$, which is Abelian, it admits two order-2-elements centralizing γ and turning the geodesic line through e onto itself such that the image of e touches v from the side opposite to e (illustration: $\overset{e \quad \gamma e}{\underset{v}{\bullet\!-\!\bullet\!-\!\bullet}}$). In the case that Γ_v is of type $\mathcal{A}_4$, it contains a normal subgroup of type $\mathcal{D}_2$ that admits again two such elements. $\square$

Let α be any torsion element in Γ. We construct the *chain of edges associated to α* as follows. Consider the edge of the reduced torsion sub-complex to which the edge stabilized by α belongs. Use corollary 22 to represent it by a chain of edges on a geodesic line. Now, α is conjugate to an element $\gamma\alpha\gamma^{-1}$ of the stabilizer of one of the edges in the chain. By lemma 24, the element $\gamma^{-1} \in \Gamma$ maps the mentioned geodesic line to the rotation axis of α. The image under γ^{-1} of the chain of edges under consideration is the desired chain associated to α. It is well defined up to translation along the rotation axis of α.

Lemma 26. *Let α be any 2-torsion element in Γ. Then the chain of edges associated to α is a fundamental domain for the action of the centralizer of α on the rotation axis of α.*

Proof. We distinguish the following two cases of how $\langle\alpha\rangle \cong \mathbb{Z}/2$ is included into Γ.

The case $\circlearrowright\!\bullet$. Suppose that there is no subgroup of type $\mathcal{D}_2$ in Γ which contains $\langle\alpha\rangle$. Then the connected component of the 2-torsion subgraph to which any edge stabilized by $\langle\alpha\rangle$ belongs, is homeomorphic to a circle. Choose a first edge e in the chain associated to α. We have an identification γ between its end and the origin of the next edge e', and by lemma 21 we can choose it such that the edge stabilizers are conjugate under γ. We can write $\Gamma_e = \langle\alpha\rangle$ and $\Gamma_{e'} = \langle\gamma\alpha\gamma^{-1}\rangle$. By lemma 24, this gives us the identification γ^{-1} from e' to an edge on the rotation axis of α, adjacent to e because of the first condition on γ. We repeat this step until we have attached an edge δe on the orbit of the first edge e, with $\delta \in \Gamma$. As δ is an isometry, the whole chain is translated by δ from the start at e to the start at δe. So the group $\langle\delta\rangle$ acts on the rotation axis with fundamental domain our chain of edges. And $\delta\alpha\delta^{-1}$ is again the rotation of order 2 around the axis of α. So, $\delta\alpha\delta^{-1} = \alpha$ and therefore $\langle\delta\rangle < C_\Gamma(\alpha)$.

The cases $\bullet\!-\!\bullet$, $\oplus$, $\circlearrowright\!-\!\bullet$, ... Suppose that there is a subgroup G of Γ of type $G \cong \mathcal{D}_2$ containing $\langle\alpha\rangle$. If there is no further inclusion $G < G' < \Gamma$ with $G' \cong \mathcal{A}_4$, let $G' := G$. Then the chain associated to α can be chosen such that one of its endpoints is stabilized by G'. The other endpoint of this chain must then lie on a different Γ-orbit, and admit as stabilizer a group H' containing $\langle\alpha\rangle$, of type $\mathcal{D}_2$ or $\mathcal{A}_4$. By lemma 25, each G' and H' contain a reflection of the rotation axis of α, centralizing α. These two reflections must differ from one another because they do not fix the chain of edges. So their free product tessellates the rotation axis of α with images of the chain of edges associated to α. $\square$

Lemma 27. *Let α be any non-trivial torsion element in a Bianchi group Γ. Then the Γ-image of the chain of edges associated to α contains the rotation axis of α.*

Proof. For α a 2-torsion element, this follows directly from lemma 26. So we can assume α to be a 3-torsion element.

The case $\bullet\!\!-\!\!\bullet$. The non-centralizing reflections of $\mathcal{S}_3$, which is associated to the two vertices, tessellate a geodesic line. Remark that there are exactly three maps $\mathbb{Z}/3 \to \mathbb{Z}/3$ induced on the edge stabilizers, given by the three order-2-elements of $\mathcal{S}_3$. So, the quotient space by the centralizer of α must be bigger.

The case $\circlearrowright$. After finitely many translations, all the edges in the fundamental domain conjugate to edges on the geodesic line are obtained as images. Hence, the Γ–image contains the whole geodesic line. $\qquad\square$

Proof of theorem 20. Corollary 23 associates a conjugacy class of type $\mathbb{Z}/\ell$ to each edge of the reduced ℓ-torsion sub-complex. Conversely, let α and $\gamma\alpha\gamma^{-1}$ be conjugate torsion elements of Γ. We want to show that they stabilize edges representing the same reduced edge. By Klein, we know that the torsion elements are elliptic and hence fix some geodesic line. Every torsion element acts as the stabilizer of a line conjugate to one passing through the Bianchi fundamental polyhedron. So, it is conjugate to one of the representative edge stabilizers. By lemma 24, we know that the line fixed by α is sent by γ to the line fixed by $\gamma\alpha\gamma^{-1}$.

By lemmata 26 and 27, the union of the Γ-images of the chain associated to α contains the whole geodesic line fixed by α (because then, as the Γ-action is cellular, any cell stabilized by $\gamma\alpha\gamma^{-1}$ admits a cell on its orbit stabilized by α). So it follows that all the edges on a geodesic line belong to the same reduced edge. $\qquad\square$

Proof of theorem 16. Comparing with lemma 18, we see that the vertex set of the ℓ-conjugacy classes graph gives precisely the bifurcation points and vertices with only one adjacent edge of the ℓ-torsion sub-complex. When passing to the reduced ℓ-torsion sub-complex, we get rid of all vertices with two adjacent edges except in the disjoint circles, see [21]. By theorem 20, the edges of the ℓ-conjugacy classes graph give the edges of the reduced ℓ-torsion sub-complex. $\qquad\square$

5. The Farrell cohomology of the Bianchi groups

In this section, we are going to prove theorem 2. In order to compare with Krämer's formulae that we evaluate in the appendix, we make use of his notations for the numbers of conjugacy classes of the five types of non-trivial finite subgroups in the Bianchi groups. His symbols for these numbers are printed in the first row of the below table, and the second row gives the symbol for the type of counted subgroup.

μ_2	μ_T	μ_3	λ_{2n}	λ_4^T	λ_4^*	λ_6^*	μ_2^-
$\mathcal{D}_2$	$\mathcal{A}_4$	$\mathcal{S}_3$	$\mathbb{Z}/n$	$\mathbb{Z}/2 \subset \mathcal{A}_4$	$\mathbb{Z}/2 \subset \mathcal{D}_2$	$\mathbb{Z}/3 \subset \mathcal{S}_3$	$\mathcal{D}_2 \nsubseteq \mathcal{A}_4$

Here, the inclusion signs "$\subset$" mean that we only consider copies of $\mathbb{Z}/n$ admitting the specified inclusion in the given Bianchi group and $\mathcal{D}_2 \nsubseteq \mathcal{A}_4$ means that we only consider copies of $\mathcal{D}_2$ not admitting any inclusion into a subgroup of type $\mathcal{A}_4$ of the Bianchi group.

Note that the number μ_2^- is simply the difference $\mu_2 - \mu_T$, because every copy of $\mathcal{A}_4$ admits precisely one normal subgroup of type $\mathcal{D}_2$. Also, note the following graph-theoretical properties

of the reduced torsion subgraphs, the latter of which we obtain by restricting our attention to the connected components not homeomorphic to ⟳.

Corollary 28 (Corollary to lemma 18)**.** *For all Bianchi groups with units $\{\pm 1\}$, the numbers of conjugacy classes of finite subgroups satisfy $\lambda_4^T \leqslant \mu_T$ and $2\lambda_6^* = \mu_3$, and even*

$$2\lambda_4^* = \mu_T + 3\mu_2^-.$$

The values given by Krämer's formulae are matching with the values computed with [22].

***Observation* 29.** The numbers of conjugacy classes of finite subgroups determine the 3-conjugacy classes graph and hence the reduced 3-torsion sub-complex for all Bianchi groups with units $\{\pm 1\}$, as we can see immediately from theorem 16 and the description of the reduced 3-torsion sub-complex in [21].

For the proof of theorem 2, we need the following ingredients.

***Remark* 30.** In the equivariant spectral spectral sequence converging to the Farrell cohomology of $\mathrm{PSL}_2\left(\mathcal{O}_{-m}\right)$, the restriction of the differential to maps between cohomology groups of cells that are not adjacent in the orbit space, are zero. So, the ℓ-primary part of the degree-1-differentials of this sequence can be decomposed as a direct sum of the blocks associated to the connected components of the ℓ-torsion sub-complex (Compare with sub-lemma 45 of [21]).

Lemma 31 (Schwermer/Vogtmann)**.** *Let M be $\mathbb{Z}$ or $\mathbb{Z}/2$. Consider group homology with trivial M-coefficients. Then the following holds.*

- *Any inclusion $\mathbb{Z}/2 \to \mathcal{S}_3$ induces an injection on homology.*
- *An inclusion $\mathbb{Z}/3 \to \mathcal{S}_3$ induces an injection on homology in degrees congruent to 3 or 0 mod 4, and is otherwise zero.*
- *Any inclusion $\mathbb{Z}/2 \to \mathcal{D}_2$ induces an injection on homology in all degrees.*
- *An inclusion $\mathbb{Z}/3 \to \mathcal{A}_4$ induces injections on homology in all degrees.*
- *An inclusion $\mathbb{Z}/2 \to \mathcal{A}_4$ induces injections on homology in degrees greater than 1, and is zero on H_1.*

For the proof in $\mathbb{Z}$-coefficients, see [25], for $\mathbb{Z}/2$-coefficients see [21].

Lemma 32 ([21], lemma 32)**.** *Let $q \geqslant 3$ be an odd integer number. Let v be a vertex representative of stabilizer type $\mathcal{D}_2$ in the refined cellular complex for the Bianchi groups. Then the three images in $(\mathrm{H}_q(\mathcal{D}_2; \mathbb{Z}))_{(2)}$ induced by the inclusions of the stabilizers of the edges adjacent to v, are linearly independent.*

Finally, we establish the following last ingredient for the proof of theorem 2, which might be of interest in its own right.

Lemma 33. *In all rows $q > 1$ and outside connected components of type ⟳, the $d_{p,q}^1$-differential of the equivariant spectral sequence converging to $\mathrm{H}_{p+q}(\mathrm{PSL}_2\left(\mathcal{O}_{-m}\right); \mathbb{Z})$ is always injective.*

Proof. For matrix blocks of the $d_{p,q}^1$-differential associated to vertices with just one adjacent edge, we see from lemma 18 that the vertex stabilizer is of type $\mathcal{A}_4$ in 2-torsion, respectively of type $\mathcal{S}_3$ in 3-torsion, so injectivity follows from lemma 31. As we have placed ourselves outside connected components of type ⟳, the remaining vertices are bifurcation points of stabilizer type $\mathcal{D}_2$ and injectivity follows from lemma 32. $\qquad\square$

Proof of theorem 2. In 3-torsion, theorem 2 follows directly from observation 29, corollary 28 and theorem 9. In 2-torsion, what we need to determine with the numbers of conjugacy classes of finite subgroups, is the 2-primary part of the $E_{p,q}^2$-term of the equivariant spectral sequence converging to $H_{p+q}(\mathrm{PSL}_2\left(\mathcal{O}_{-m}\right); \mathbb{Z})$ in all rows $q > 1$. From there, we see from theorem 9 that we obtain the claim. By remark 30, we only need to check this determination on each homeomorphism type of connected components of the 2-torsion subgraph. We use theorem 16 to identify the reduced 2-torsion subgraph and the 2-conjugacy classes graph. Then we can observe that

- Krämer's number $\lambda_4^* - \lambda_4$ determines the number of connected components of type ⟳.
- Krämer's number λ_4^* determines the number of edges of the 2-torsion subgraph outside connected components of type ⟳. Lemma 33 tells us that the block of the $d_{p,q}^1$-differential of the equivariant spectral sequence associated to such edges is always injective.
- Krämer's number μ_2^- determines the number of bifurcation points, and μ_T determines the number of vertices with only one adjacent edge of the 2-torsion subgraph.

Using corollary 28, we obtain the explicit formulae in theorem 2.   □

6. THE COHOMOLOGY RING STRUCTURE OF THE BIANCHI GROUPS

In [5], Berkove has found a compatibility of the cup product of the cohomology ring of a Bianchi group with the cup product of the cohomology rings of its finite subgroups. This compatibility within the equivariant spectral sequence implies that all products that come from different connected components of the reduced torsion sub-complex (which we turn into the conjugacy classes graph in section 4) are zero. It follows that the cohomology ring of any Bianchi group splits into a restricted sum over sub-rings, which depend in degrees above the virtual cohomological dimension only on the homeomorphism type of the associated connected component of the reduced torsion sub-complex. The analogue in cohomology of theorem 2 and Berkove's computations of sample cohomology rings [4] yield the following corollary in 3-torsion.

We use Berkove's notation, in which the degree j of a cohomology generator x_j is appended as a subscript. Furthermore, writing cohomology classes inside square brackets means that they are polynomial (of infinite multiplicative order), and writing them inside parentheses means that they are exterior (their powers vanish). The restricted sum $\widetilde{\oplus}$ identifies all the degree zero classes into a single copy of $\mathbb{Z}$; when we write it with a power, we specify the number of summands. Recall that λ_6 (respectively μ_3) counts the number of conjugacy classes of subgroups of type $\mathbb{Z}/3$ (respectively $\mathcal{S}_3$) in the Bianchi group.

Corollary 34. *In degrees above the virtual cohomological dimension, the* 3-*primary part of the cohomology ring of any Bianchi group* Γ *with units* $\{\pm 1\}$ *is given by*

$$\mathrm{H}^*(\Gamma; \mathbb{Z})_{(3)} \cong \widetilde{\oplus}^{(\lambda_6 - \frac{\mu_3}{2})}\mathbb{Z}[x_2](\sigma_1) \,\widetilde{\oplus}^{\frac{\mu_3}{2}}\mathbb{Z}[x_4](x_3),$$

where the generators x_j *are of additive order* 3.

In 2-torsion, it does in general not suffice to know only the numbers of conjugacy classes of finite subgroups to obtain the cohomology ring structure, because for the two reduced 2-torsion sub-complexes ⟳• ⟳• and ⬯ •—•, we obtain the same numbers of conjugacy classes and homological 2-torsion, but different multiplicative structures of the mod-2 cohomology rings, as we can see from table 1, which we compile from the results of [5] (and [21]).

T	subring associated to connected components of type T in $2-$ conjugacy classes graph
$\circlearrowright$	$\mathbb{F}_2[n_1](m_1)$
•—•	$\mathbb{F}_2[m_3, u_2, v_3, w_3]/\langle m_3v_3 = 0, \quad u_2^3 + w_3^2 + v_3^2 + m_3^2 + w_3(v_3 + m_3) = 0\rangle$
$\circlearrowright$—•	$\mathbb{F}_2[n_1, m_2, n_3, m_3]/\langle n_1n_3 = 0, \quad m_2^3 + m_3^2 + n_3^2 + m_3n_3 + n_1m_2m_3 = 0\rangle$
$\oplus$	$\mathbb{F}_2[n_1, m_1, m_3]/\langle m_3(m_3 + n_1^2m_1 + n_1m_1^2) = 0\rangle$

TABLE 1. Restricted summands of the mod-2 cohomology ring $\mathrm{H}^*(\Gamma; \mathbb{F}_2)$ of a Bianchi group Γ above its virtual cohomological dimension.

***Observation* 35.** In the cases of class numbers 1 and 2, only the homeomorphism types T listed in table 1 occur as connected components in the reduced 2-torsion sub-complex. So for all such Bianchi groups Γ with units $\{\pm 1\}$, the mod-2 cohomology ring $\mathrm{H}^*(\Gamma; \mathbb{F}_2)$ splits, above the virtual cohomological dimension, as a restricted sum over the sub-rings specified in table 1, with powers according to the multiplicities of the occurrences of the types T.

Appendix A. Numerical evaluation of Krämer's formulae

A.1. **Numbers of conjugacy classes in 3-torsion.** Denote by δ the number of finite ramification places of $\mathbb{Q}(\sqrt{-m})$ over $\mathbb{Q}$. Let k_+ be the totally real number field $\mathbb{Q}(\sqrt{3m})$ and denote its ideal class number by h_{k_+}. Krämer introduces the following indicators:

$$z := \begin{cases} 2, & \text{if 3 is the norm of an integer of } k_+, \\ 1, & \text{otherwise.} \end{cases}$$

For $m \equiv 0 \mod 3$ and $m \neq 3$, denote by $\epsilon := \frac{1}{2}(a + b\sqrt{\frac{m}{3}}) > 1$ the fundamental unit of k_+ (where $a, b \in \mathbb{N}$). Now, define

$$x' := \begin{cases} 2, & \text{if the norm of } \epsilon \text{ is } 1, \\ 1, & \text{if the norm of } \epsilon \text{ is } -1 \end{cases}$$

and

$$y := \begin{cases} 2, & \text{if } b \equiv 0 \mod 3, \\ 1, & \text{otherwise.} \end{cases}$$

Then [16, 20.39 and 20.41] yield the following formulae in 3-torsion.

m specifying Bianchi groups $\mathrm{PSL}_2(\mathcal{O}_{-m})$	λ_6^*	$\lambda_6 - \lambda_6^*$
$m \equiv 2 \mod 3$	0	$\frac{z}{2}h_{k_+}$
$m \equiv 1 \mod 3$ gives either	$2^{\delta-1}$	$\frac{1}{2}(h_{k_+} - 2^{\delta-1})$
or	0	$\frac{1}{2}h_{k_+}$
$m \equiv 6 \mod 9$	0	$x'yh_{k_+}$
$m \equiv 3 \mod 9,\ m \neq 3$ gives either	$2^{\delta-2}$	$\frac{1}{2}(3x'h_{k_+} - 2^{\delta-2})$
or	0	$\frac{1}{2}3x'h_{k_+}$

The above case distinctions come from the fact that Krämer's theorem 20.39 ranges over all types of maximal orders in quaternion algebras over $\mathbb{Q}(\sqrt{-m})$, in which Krämer determines the numbers of conjugacy classes in the norm-1-group. The remaining task in order to decide which of the cases applies, is to find out of which type considered in the mentioned theorem is the maximal order $\mathrm{M}_2(\mathcal{O}_{-m})$. Some methods to cope with this task are introduced in [16, §27].

Krämer's resulting criteria can be summarized as follows for 3-torsion.

condition	implication
$m \equiv 2 \mod 3$	$\mu_3 = \lambda_6^* = 0.$
$m \equiv 6 \mod 9$	$\mu_3 = \lambda_6^* = 0.$
m prime and $m \equiv 1 \mod 3$	$\lambda_6^* > 0.$
$m = 3p$ with p prime and $p \equiv 1 \mod 3$	$\lambda_6^* > 0.$
$m \equiv 1 \mod 3$ and -3 occurs as norm on $\mathcal{O}_{k_+}$	$\lambda_6^* > 0.$
$m \equiv 1 \mod 3$ and -3 does not occur as norm on $\mathcal{O}_{k_+}$	$\lambda_6 - \lambda_6^* > 0.$
$m \equiv 1 \mod 3$ and m admits a prime divisor p with $p \equiv 2 \mod 3$	$\lambda_6 - \lambda_6^* > 0.$
$m \equiv 3 \mod 9$ and $x' = 1$ and $\frac{m}{3}$ admits only prime divisors p with $p \equiv 1 \mod 12$	$\lambda_6^* > 0.$
$m \equiv 3 \mod 9$ and $x' = 1$ and $\frac{m}{3}$ admits a prime divisor p with $p \equiv 5 \mod 12$	$\lambda_6^* = 0.$
$m \equiv 3 \mod 9$ and $h(k'_+) = 2^{\delta-3}$ and $\frac{m}{3}$ admits only prime divisors p with $p \equiv \pm 1 \mod 12$ or $p = 2$	$\lambda_6^* = 0.$
$m \equiv 3 \mod 9$ and $h(k'_+) = 1$ and $\frac{m}{3} = p'p$ with p', p prime and $p' \equiv p \equiv 7 \mod 12$	$\lambda_6^* > 0.$

In order to determine Krämer's indicator z, we need to determine if a given value occurs as the norm on the ring of integers of an imaginary quadratic number field. This is implemented in Pari/GP [1] (the first step is computing the answer under the Generalized Riemann hypothesis, and the second step is a check computation which confirms that we arrive at that answer without this hypothesis). Additionally, we compare with the below criterion [16, (20.13)].

Lemma 36 (Krämer). *Let m be not divisible by* 3.

- *If the number -3 is the norm of an integer in the totally real number field k_+, then all prime divisors $p \in \mathbb{N}$ of m satisfy the congruence $p \equiv 1 \mod 3$.*
  Especially, the congruence $m \equiv 1 \mod 3$ is implied.
- *If the number* 3 *is the norm of an integer in the totally real number field k_+, then all prime divisors $p \in \mathbb{N}$ of m satisfy either $p = 2$ or the congruence $p \equiv \pm 1 \mod 12$.*
 *Additionally, the congruence $m \equiv 2 \mod 3$ is implied.*

With Krämer's criteria at hand, we can decide for many Bianchi groups, which of the alternative cases in Krämer's formulae must be used. We do this in the below tables for all such Bianchi groups $\mathrm{PSL}_2\left(\mathcal{O}_{-m}\right)$ with absolute value of the discriminant Δ ranging between 7 and 2003, where we recall that the discriminant is $\Delta = \begin{cases} -m, & m \equiv 3 \mod 4, \\ -4m, & \text{else.} \end{cases}$

In the cases $m \in \{102, 133, 165, 259, 559, 595, 763, 835, 1435\}$, where these statements are not sufficient to eliminate the wrong alternatives, we insert the results of [22]. This way, the below tables treat all Bianchi groups with units $\{\pm 1\}$ and discriminant of absolute value less than 615. The cases where an ambiguity remains (so to exclude them from our tables) are $m \in \{210, 262, 273, 298, 345, 426, 430, 462, 481, 615, 1155, 1159, 1195, 1339, 1351, 1407, 1515, 1807\}$. For tables of the cases without ambiguity, with m ranging up to 10000, see the preprint version 2 of this paper on HAL.

In [15], a theorem is established which solves all these ambiguities by giving for each type of finite subgroups in $\mathrm{PSL}_2\left(\mathcal{O}_{-m}\right)$ criteria equivalent to its occurrence, in terms of congruence conditions on the prime divisors of m.

3−conjugacy classes graph	m specifying Bianchi groups $\mathrm{PSL}_2\left(\mathcal{O}_{-m}\right)$ with this 3−conjugacy classes graph
⟳	2, 5, 6, 10, 11, 14, 15, 17, 22, 23, 29, 34, 35, 38, 41, 46, 47, 51, 53, 55, 58, 59, 62, 71, 82, 83, 86, 87, 89, 94, 95, 101, 106, 113, 115, 118, 119, 123, 131, 134, 137, 142, 149, 155, 158, 159, 166, 167, 173, 178, 179, 187, 191, 197, 202, 203, 206, 214, 215, 226, 227, 233, 235, 239, 251, 254, 257, 263, 267, 269, 274, 278, 281, 287, 293, 295, 303, 311, 317, 319, 323, 326, 334, 335, 339, 346, 347, 353, 355, 358, 359, 371, 382, 383, 389, 391, 394, 395, 398, 401, 411, 415, 422, 431, 443, 446, 447, 449, 451, 454, 461, 466, 467, 478, 479, 491, 515, 519, 527, 535, 551, 563, 583, 591, 599, 623, 635, 647, 655, 659, 667, 683, 695, 699, 707, 719, 731, 743, 755, 779, 791, 799, 807, 815, 827, 839, 843, 879, 887, 895, 899, 911, 943, 947, 951, 955, 959, 979, 983, 995, 1003, 1019, 1031, 1055, 1059, 1091, 1103, 1111, 1115, 1135, 1139, 1151, 1163, 1167, 1187, 1207, 1211, 1219, 1223, 1243, 1247, 1255, 1259, 1271, 1283, 1307, 1315, 1343, 1347, 1363, 1367, 1379, 1383, 1411, 1415, 1439, 1487, 1499, 1507, 1511, 1523, 1527, 1535, 1555, 1559, 1563, 1571, 1607, 1631, 1639, 1643, 1655, 1667, 1671, 1707, 1711, 1735, 1751, 1763, 1779, 1787, 1795, 1799, 1811, 1819, 1823, 1835, 1847, 1851, 1883, 1903, 1907, 1915, 1919, 1923, 1927, 1931, 1943, 1959, 1979, 2003,
2 ⟳	26, 42, 65, 69, 70, 74, 77, 78, 85, 110, 122, 130, 141, 143, 145, 154, 161, 170, 182, 185, 186, 190, 194, 195, 205, 209, 213, 218, 221, 222, 230, 231, 238, 253, 265, 266, 286, 305, 310, 314, 322, 329, 365, 366, 370, 377, 386, 406, 407, 410, 418, 434, 437, 442, 445, 455, 458, 470, 473, 474, 483, 485, 493, 494, 497, 555, 611, 627, 671, 715, 767, 803, 851, 923, 935, 1015, 1079, 1095, 1199, 1235, 1295, 1311, 1391, 1403, 1455, 1463, 1491, 1495, 1595, 1599, 1615, 1679, 1703, 1739, 1771, 1855, 1887, 1991,
3 ⟳	30, 66, 107, 138, 174, 255, 282, 302, 318, 354, 419, 498, 503, 759, 771, 795, 835, 863, 1007, 1319, 1355, 1427, 1479, 1551, 1583, 1619, 1691, 1695, 1871, 1895, 1947, 1967,
4 ⟳	33, 105, 114, 146, 177, 249, 258, 285, 290, 299, 321, 330, 341, 357, 374, 385, 393, 402, 413, 429, 465, 482, 595, 663, 915, 987, 1023, 1067, 1239, 1435, 1727, 1743, 1955, 1995,
5 ⟳	1043, 1203, 1451,
6 ⟳	102, 165, 246, 362, 390, 435, 1335, 1419, 1547,
7 ⟳	587, 971,

3−conjugacy classes graph	m specifying Bianchi groups $\mathrm{PSL}_2\left(\mathcal{O}_{-m}\right)$ with this 3−conjugacy classes graph
8 ↻•	438, 1131, 1635,
•—•	7, 19, 31, 43, 67, 79, 103, 127, 139, 151, 163, 199, 211, 223, 271, 283, 307, 379, 439, 463, 487, 499, 523, 571, 607, 619, 631, 691, 727, 739, 751, 787, 811, 823, 859, 883, 907, 919, 967, 991, 1039, 1051, 1063, 1123, 1171, 1231, 1279, 1303, 1399, 1423, 1447, 1459, 1471, 1483, 1531, 1543, 1567, 1579, 1627, 1663, 1699, 1723, 1759, 1783, 1831, 1867, 1987, 1999,
•—• ⊔ ↻•	39, 111, 183, 219, 291, 327, 331, 367, 471, 543, 579, 643, 723, 831, 939, 1011, 1047, 1087, 1119, 1191, 1227, 1263, 1291, 1299, 1327, 1371, 1623, 1803, 1839, 1879, 1951, 1983,
•—• ⊔ 2 ↻•	547, 1747,
•—• ⊔ 4 ↻•	687,
•—• ⊔ 10 ↻•	1731,
2 •—•	13, 37, 61, 91, 109, 157, 181, 229, 247, 277, 349, 373, 403, 421, 427, 511, 679, 703, 871, 1099, 1147, 1267, 1591, 1603, 1687, 1891, 1963,
2 •—• ⊔ ↻•	73, 97, 193, 241, 259, 313, 337, 409, 457, 559, 763, 1651, 1939,
2 •—• ⊔ 2 ↻•	21, 57, 93, 129, 201, 309, 381, 397, 399, 417, 453, 489, 651, 903, 1443, 1659, 1767, 1843,
2 •—• ⊔ 3 ↻•	433, 1027, 1387,
2 •—• ⊔ 8 ↻•	237,
4 •—•	217, 301, 469,
4 •—• ⊔ 2 ↻•	133.

A.2. Numbers of conjugacy classes in 2-torsion.

Denote by δ the number of finite ramification places of $\mathbb{Q}(\sqrt{-m})$ over $\mathbb{Q}$. Let k_+ be the totally real number field $\mathbb{Q}(\sqrt{m})$ and denote its ideal class number by h_{k_+}. For $m \neq 1$, Krämer introduces the following indicators:

$$z := \begin{cases} 2, & \text{if 2 is the norm of an integer of } k_+, \\ 1, & \text{otherwise,} \end{cases} \qquad q := \begin{cases} 2, & \text{if } \pm 2 \text{ is the norm of an integer of } k_+, \\ 1, & \text{otherwise,} \end{cases}$$

$$w := \begin{cases} 2, & \text{if } \forall \text{ prime divisors } p \text{ of } m \text{ with } p \neq 2 \text{ we have } p \equiv \pm 1 \mod 8, \\ 1, & \text{if } m \text{ admits prime divisors } p \equiv \pm 3 \mod 8. \end{cases}$$

Furthermore, denote by $\epsilon := \frac{1}{2}(a + b\sqrt{m}) > 1$ the fundamental unit of k_+ (where $a, b \in \mathbb{N}$). Now, define

$$x := \begin{cases} 2, & \text{if the norm of } \epsilon \text{ is 1,} \\ 1, & \text{if the norm of } \epsilon \text{ is } -1 \end{cases} \qquad \text{and} \qquad y := \begin{cases} 3, & \text{if } b \equiv 0 \mod 2, \\ 1, & \text{if } b \equiv 1 \mod 2. \end{cases}$$

Then [16, 26.12 and 26.14] yield the following formulae in 2-torsion.

m specifying Bianchi groups $\mathrm{PSL}_2\left(\mathcal{O}_{-m}\right)$	μ_T	μ_2^-	λ_4^T	λ_4^*	$\lambda_4 - \lambda_4^*$
$m \equiv 7 \mod 8$	0	0	0	0	$\frac{z}{2} h_{k_+}$
$m \equiv 3 \mod 8$ gives either	2^δ	0	$2^{\delta-1}$	$2^{\delta-1}$	$\frac{1}{2}(h_{k_+} - 2^{\delta-1})$
or (provided that $2^{\delta-1} > 1$)	0	0	0	0	$\frac{1}{2} h_{k_+}$
$m \equiv 2 \mod 4$ and $w = 2$ gives either	$2^{\delta-1}$	$2^{\delta-1}$	$2^{\delta-2}z$	2^δ	$\frac{1}{4}x(z+2)h_{k_+} - 2^{\delta-1}$
or (provided that $2^{\delta-1} > 1$)	0	0	0	0	$\frac{1}{4}x(z+2)h_{k_+}$
$m \equiv 2 \mod 4$ and $w = 1$ gives either	$2^{\delta-1}$	0	$2^{\delta-2}$	$2^{\delta-2}$	$\frac{1}{2}(\frac{3}{2}xh_{k_+} - 2^{\delta-2})$
or	0	$2^{\delta-1}$	0	$2^{\delta-2}3$	$\frac{3}{2}(\frac{1}{2}xh_{k_+} - 2^{\delta-2})$
or (provided that $2^{\delta-1} > 2$)	0	0	0	0	$\frac{3}{4}xh_{k_+}$
$m \equiv 1 \mod 8$ and $m \neq 1$ and $w = 2$ gives either	$2^{\delta-1}$	$2^{\delta-1}$	$2^{\delta-2}$	2^δ	$2xh_{k_+} - 2^{\delta-1}$
or (provided that $2^{\delta-2} > 1$)	0	0	0	0	$2xh_{k_+}$
$m \equiv 1 \mod 8$ and $w = 1$ gives either	$2^{\delta-1}$	0	$2^{\delta-2}$	$2^{\delta-2}$	$2xh_{k_+} - 2^{\delta-3}$
or	0	$2^{\delta-1}$	0	$2^{\delta-2}3$	$2xh_{k_+} - 2^{\delta-3}3$
or (provided that $2^{\delta-2} > 2$)	0	0	0	0	$2xh_{k_+}$
$m \equiv 5 \mod 8$	0	$2^{\delta-1}$	0	$2^{\delta-2}3$	$\frac{1}{2}\left(x(2y+1)h_{k_+} - 2^{\delta-2}3\right)$
or (provided that $2^{\delta-2} > 1$)	0	0	0	0	$\frac{1}{2}x(2y+1)h_{k_+}$

The above case distinctions come from the fact that Krämer's theorem 26.12 ranges over all types of maximal orders in quaternion algebras over $\mathbb{Q}(\sqrt{-m})$, in which Krämer determines the numbers of conjugacy classes in the norm-1-group. The remaining task in order to decide which of the cases applies, is to find out of which type considered in the mentioned theorem is the maximal order $\mathrm{M}_2(\mathcal{O}_{-m})$. Some methods to cope with this task are introduced in [16, §27], where Krämer obtains the following criteria for the 2-torsion numbers:

| condition	implication
$m \equiv 7 \mod 8$	$\mu_T = \mu_2^- = \lambda_4^T = \lambda_4^* = 0.$
$m \equiv 5 \mod 8$	$\mu_T = \lambda_4^T = 0.$
$m \equiv 21 \mod 24$	$\lambda_4^* = 0.$
$m \equiv 0 \mod 6$ and $\lambda_4^* > 0$	$\lambda_4^T > 0.$
$m \equiv 9 \mod 24$ and $\lambda_4^* > 0$	$\lambda_4^T > 0.$
m prime and $m \equiv 1$ or $3 \mod 8$	$\lambda_4^T > 0.$
$m \equiv 5 \mod 8$ and m prime	$\lambda_4^* > 0.$
$m = 2p$ with p prime and $p \equiv 3$ or $5 \mod 8$	$\lambda_4^* > 0.$
$m = p'p$ with p and p' prime and $p \equiv p' \equiv 3$ or $5 \mod 8$	$\lambda_4^* > 0.$
$m = 3p$ with p prime and $p \equiv 1$ or $3 \mod 8$	$\lambda_4^T > 0.$
$m \equiv 1$ or $2 \mod 4$ and $m \neq 1$ and $x = 1$	$\lambda_4^* > 0$ and $\mu_2^- > 0.$
$m \equiv 1$ or $2 \mod 4$ and $m \neq 1$ and $x = 2$	$\lambda_4 - \lambda_4^* > 0.$
$m \equiv 3 \mod 8$ and -2 occurs as norm on $\mathcal{O}_{k_+}$	$\lambda_4^* > 0$ and $\lambda_4^T > 0.$
$m \equiv 3 \mod 8$ and -2 does not occur as norm on $\mathcal{O}_{k_+}$	$\lambda_4 - \lambda_4^* > 0.$
$m \equiv 3 \mod 8$ and m admits a prime divisor p with $p \equiv 5$ or $7 \mod 8$	$\lambda_4 - \lambda_4^* > 0.$
$m \equiv 1 \mod 8$ and $w = 1$ and $h(k_+) = 2^{\delta-3}$	$\mu_2^- = 0.$
$m \equiv 2 \mod 4$ and -2 occurs as norm on $\mathcal{O}_{k_+}$	$\lambda_4^T > 0.$
$m \equiv 2 \mod 4$ and -2 does not occur as norm on $\mathcal{O}_{k_+}$ and $h(k_+) = 2^{\delta-2}$	$\lambda_4^* = 0.$
$m \equiv 2 \mod 4$ and $q = 1$ and $h(k_+) = 2^{\delta-1}$ and $w = 2$	$\lambda_4^* = 0.$
$m \equiv 2 \mod 4$ and $h(k_+) = 2^{\delta-2}$ and m admits a prime divisor p with $p \equiv 5$ or $7 \mod 8$	$\lambda_4^* = 0.$

With the above criteria at hand, we can decide for many Bianchi groups, which of the alternative cases in Krämer's formulae must be used. We do this in the below tables for all such Bianchi groups $\mathrm{PSL}_2(\mathcal{O}_{-m})$ with absolute value of the discriminant Δ ranging between 7 and 2003. In the cases $m \in \{\, 34, 105, 141, 142, 194, 235, 323, 427, 899, 979, 1243, 1507\}$, where these statements are not sufficient to eliminate the wrong alternatives, we insert the results of [22]. This way, the below tables treat all Bianchi groups with units $\{\pm 1\}$ and discriminant of absolute value less than 820. The cases where an ambiguity remains (so to exclude them from our tables) are the following values of m: 205, 221, 254, 273, 305, 321, 322, 326, 345, 377, 381, 385, 386, 410, 438, 465, 469, 473, 482, 1067, 1139, 1211, 1339, 1443, 1763, 1771, 1947.

The above mentioned theorem on subgroup occurrences [15] solves all these ambiguities.

2−torsion homology	m specifying Bianchi groups $\mathrm{PSL}_2(\mathcal{O}_{-m})$ with this 2−torsion homology
P	7, 15, 23, 31, 35, 39, 47, 55, 71, 87, 91, 95, 103, 111, 115, 127, 143, 151, 155, 159, 167, 183, 191, 199, 203, 215, 239, 247, 259, 263, 271, 295, 299, 303, 311, 319, 327, 335, 355, 367, 371, 383, 395, 403, 407, 415, 431, 447, 463, 471, 479, 487, 503, 515, 519, 535, 543, 551, 559, 583, 591, 599, 607, 611, 631, 635, 647, 655, 667, 671, 687, 695, 703, 707, 719, 743, 751, 755, 763, 767, 807, 815, 823, 831, 835, 851, 863, 871, 879, 887, 911, 919, 923, 951, 955, 967, 983, 991, 995, 1007, 1027, 1031, 1039, 1043, 1047, 1055, 1063, 1079, 1099, 1103, 1115, 1119, 1135, 1147, 1151, 1159, 1167, 1195, 1199, 1219, 1231, 1247, 1255, 1263, 1267, 1279, 1303, 1315, 1319, 1355, 1363, 1379, 1383, 1391, 1399, 1403, 1415, 1423, 1439, 1447, 1471, 1487, 1511, 1535, 1543, 1555, 1559, 1583, 1591, 1603, 1607, 1623, 1643, 1651, 1655, 1663, 1671, 1703, 1711, 1727, 1739, 1759, 1783, 1795, 1807, 1823, 1831, 1835, 1839, 1871, 1879, 1883, 1891, 1895, 1903, 1915, 1919, 1939, 1943, 1951, 1959, 1963, 1983, 1991, 1999,
$2P$	14, 46, 62, 94, 119, 158, 195, 206, 231, 255, 287, 302, 334, 382, 391, 398, 435, 446, 455, 478, 483, 511, 527, 555, 595, 615, 623, 651, 663, 679, 715, 759, 791, 795, 903, 915, 935, 943, 987, 1015, 1095, 1131, 1207, 1235, 1271, 1295, 1311, 1335, 1343, 1407, 1435, 1455, 1463, 1479, 1491, 1515, 1547, 1551, 1595, 1615, 1631, 1635, 1659, 1687, 1695, 1751, 1767, 1799, 1855, 1887, 1927, 1955, 1967,
$3P$	21, 30, 42, 69, 70, 77, 78, 79, 93, 110, 133, 138, 154, 174, 182, 186, 190, 213, 222, 223, 230, 235, 237, 253, 266, 282, 286, 301, 309, 310, 318, 341, 359, 366, 406, 413, 426, 427, 430, 437, 453, 470, 474, 494, 839, 895, 899, 1191, 1223, 1367, 1527, 1567, 1639, 1735, 1847,
$4P$	161, 217, 238, 329, 399, 497, 799, 959, 1023, 1155, 1239, 1351, 1679, 1743, 1995,
$5P$	439, 727, 1111, 1327,
$6P$	142, 165, 210, 285, 330, 357, 390, 429, 434, 462, 1495, 1599,
$7P$	141, 1087,
$8P$	105,
$2P^*_{\mathcal{D}_2}$	5, 10, 13, 26, 29, 53, 58, 61, 74, 106, 109, 122, 149, 157, 173, 181, 202, 218, 277, 293, 298, 314, 317, 362, 394, 397, 421, 458, 461,

2−torsion homology	m specifying Bianchi groups $\mathrm{PSL}_2\left(\mathcal{O}_{-m}\right)$ with this 2−torsion homology
$2P^*_{\mathcal{D}_2} + 2P_{\bigcirc}$	37, 101, 197, 269, 349, 373, 389,
$2P^*_{\mathcal{D}_2} + 3P_{\bigcirc}$	229, 346,
$4P^*_{\mathcal{D}_2}$	85, 130, 170, 290, 365, 370, 493,
$4P^*_{\mathcal{D}_2} + P_{\bigcirc}$	65, 185, 265, 481,
$4P^*_{\mathcal{D}_2} + 3P_{\bigcirc}$	442, 445,
$4P^*_{\mathcal{D}_2} + 4P_{\bigcirc}$	485,
$4P^*_{\mathcal{D}_2} + 5P_{\bigcirc}$	145,
$P^*_{\mathcal{A}_4} + P^*_{\mathcal{D}_2}$	2,
$2P^*_{\mathcal{A}_4}$	11, 19, 43, 59, 67, 83, 107, 131, 139, 163, 179, 211, 227, 251, 283, 307, 331, 347, 379, 419, 467, 491, 523, 547, 563, 571, 587, 619, 643, 683, 691, 739, 787, 811, 827, 859, 883, 907, 947, 971, 1019, 1051, 1123, 1163, 1187, 1259, 1283, 1291, 1307, 1427, 1451, 1459, 1483, 1499, 1531, 1571, 1579, 1619, 1667, 1699, 1723, 1747, 1867, 1931, 1979, 2003,
$2P^*_{\mathcal{A}_4} + P_{\bigcirc}$	6, 22, 38, 86, 118, 134, 166, 214, 262, 278, 358, 422, 443, 454, 659, 1091, 1171, 1523, 1627, 1787, 1811, 1907, 1987,
$2P^*_{\mathcal{A}_4} + 2P_{\bigcirc}$	499,
$2P^*_{\mathcal{A}_4} + 2P^*_{\mathcal{D}_2}$	17, 41, 73, 89, 97, 113, 137, 193, 233, 241, 281, 313, 337, 353, 409, 433, 449, 457,
$2P^*_{\mathcal{A}_4} + 2P^*_{\mathcal{D}_2} + P_{\bigcirc}$	82, 146, 178, 274, 466,
$2P^*_{\mathcal{A}_4} + 2P^*_{\mathcal{D}_2} + 2P_{\bigcirc}$	34, 194,
$2P^*_{\mathcal{A}_4} + 2P^*_{\mathcal{D}_2} + 4P_{\bigcirc}$	226, 257,
$2P^*_{\mathcal{A}_4} + 2P^*_{\mathcal{D}_2} + 8P_{\bigcirc}$	401,
$4P^*_{\mathcal{A}_4}$	51, 123, 187, 267, 339, 411, 451, 699, 771, 779, 803, 843, 1059, 1203, 1347, 1563, 1691, 1707, 1779, 1819, 1843, 1923,
$4P^*_{\mathcal{A}_4} + P_{\bigcirc}$	219, 291, 323, 579, 723, 731, 939, 979, 1003, 1011, 1227, 1243, 1371, 1387, 1411, 1507, 1731, 1803,
$4P^*_{\mathcal{A}_4} + 2P_{\bigcirc}$	66, 102, 114, 246, 258, 354, 374, 402, 418, 498, 1851,
$4P^*_{\mathcal{A}_4} + 3P_{\bigcirc}$	33, 57, 129, 177, 201, 209, 249, 393, 417, 489, 1299,
$8P^*_{\mathcal{A}_4}$	627, 1419.

FIGURE 6. Average homological 3-torsion outside subgroups of type $\mathcal{S}_3$, scaled as indicated.
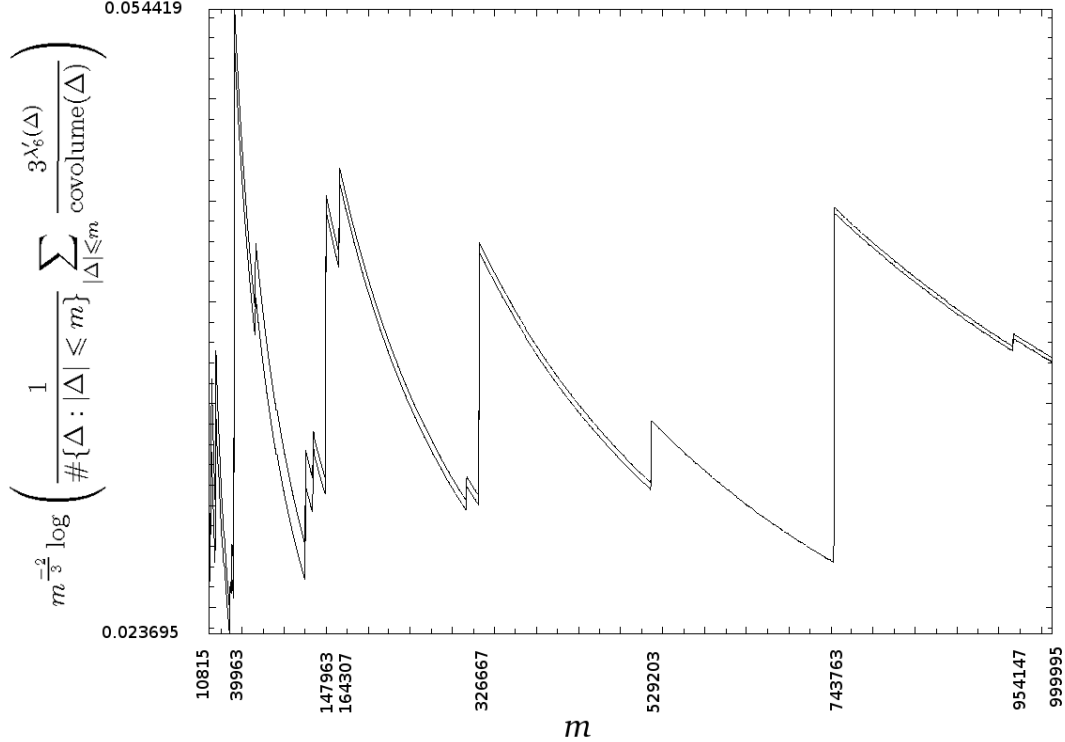
A.3. Asymptotic behavior of the number of conjugacy classes.

From Krämer's above formulae, we see that both in 2-torsion and in 3-torsion, the number of conjugacy classes of finite subgroups, and hence also the cardinality of the homology of the Bianchi groups in degrees above their virtual cohomological dimension, admits only two factors which are not strictly limited: h_{k_+} and 2^δ. As for the ideal class number h_{k_+}, it is subject to the predictions of the Cohen-Lenstra heuristic [8]. As for the factor 2^δ, the number δ of finite ramification places of $\mathbb{Q}(\sqrt{-m})$ over $\mathbb{Q}$ is well-known to equal the number of prime divisors of the discriminant of $\mathbb{Q}(\sqrt{-m})$.

The numerical evaluation of Krämer's formulae provides us with databases which are over a thousand times larger than what is reasonable to print in sections A.1 and A.2. We now give an instance of how these databases can be exploited. Denote the discriminant of $\mathbb{Q}(\sqrt{-m})$ by Δ. In the cases $m \equiv 3 \mod 4$, we have $\Delta = -m$. Denote the number $\lambda_6 - \lambda_6^*$ of connected components of type ◯ in the 3-conjugacy classes graph by $\lambda_6'(\Delta)$. Then clearly, the subgroup in $H_q(\mathrm{PSL}_2(\mathcal{O}_{-m}))$, $q > 2$, generated by the order-3-elements coming from the connected components of this type, is of order $3^{\lambda_6'(\Delta)}$. Denote by covolume(Δ) the volume of the quotient space $\mathrm{PSL}_2(\mathcal{O}_{-m})\backslash\mathcal{H}$. The study of the ratio $\frac{3^{\lambda_6'(\Delta)}}{\text{covolume}(\Delta)}$ is motivated by the formulae in [3]. In figure 6, we print the logarithm of the average of this ratio over the cases $|\Delta| \equiv 3 \mod 4$, scaled by a

factor $m^{\frac{-2}{3}}$, so to say

$$m^{\frac{-2}{3}} \log \left(\frac{1}{\#\{\Delta : |\Delta| \leqslant m\}} \sum_{|\Delta| \leqslant m} \frac{3^{\lambda'_6(\Delta)}}{\text{covolume}(\Delta)} \right),$$

where we consider m and Δ as independent variables, m running through the square-free positive rational integers. In order to cope with the fact that in some cases, Krämer's formulae leave an ambiguity, we print a function assuming the lowest possible values of $\lambda'_6(\Delta)$ and one assuming the highest possible values of $\lambda'_6(\Delta)$ in the same diagram.

So for m greater than 10815 and less than one million, we can observe that the average of the above ratio oscillates between $\exp(m^{\frac{2}{3}}0.023695)$ and $\exp(m^{\frac{2}{3}}0.054419)$. For m less than 10815, this oscillation is much stronger, and the diagram might be seen as suggesting that possibly the oscillation could remain between these two bounds for m greater than one million.

For related asymptotics, see the recent works of Bergeron/Venkatesh [3] and Sengün [26]. For an alternative computer program treating the Bianchi groups, see the SAGE package of Cremona's student Aranes [2], and for $\text{GL}_2(\mathcal{O})$ see Yasaki's program [31].

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